The Tschebychev polynomials (of the 1st kind) $T_n(x)$ are defined through the recurrence relation

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+2}(x) = 2x \cdot T_{n+1}(x) - T_n(x)$$

for $n \geq 0$.

Theorem:

$T_n(x)$ satisfies the differential equation $(x^2-1)\cdot T_n^{''}(x) + x \cdot T_n^{'}(x) - n^2 \cdot T_n(x) = 0$ for all $n \geq 0$.

Proof:

Let $F := \sum_{n=0}^{\infty} T_n(x) z^n$ be the generating function of the sequence of polynomials. The recurrence is equivalent to the equation

$$(1 - 2x \cdot z + z^2) \cdot F = 1 - x \cdot z$$

Proof 1:

F is a rational function in z, $F = \frac{1-xz}{1-2xz+z^2}$. From the theory of recursions with constant coefficients, we know that we have to perform a partial fraction decomposition. So let $p(z) = z^2 - 2x \cdot z + 1$ be the denominator and $\alpha = x + \sqrt{x^2-1}$ and α^{-1} its zeroes. The partial fraction decomposition reads

$$F = \frac{1-xz}{1-2xz+z^2} = \frac{1}{2}\left(\frac{1}{1-\alpha z} + \frac{1}{1-\alpha^{-1}z}\right)$$

hence $T_n(x) = \frac{1}{2}(\alpha^n + \alpha^{-n})$. Note that the field $Q(x)(\alpha)$, being a finite dimensional extension field of $Q(x)$ in characteristic 0, has a unique derivation extending $\frac{d}{dx}$ on $Q(x)$. We can therefore try to construct an annihilating differential operator for $T_n(x)$ by combination of annihilating differential operators for α^n and α^{-n}. In fact, $L_1 := (\alpha - x)\frac{d}{dx} - n$ satisfies $L_1[\alpha^n] = 0$, and $L_2 := (\alpha - x)\frac{d}{dx} + n$ satisfies $L_2[\alpha^{-n}] = 0$. A common multiple of L_1 and L_2 is easily found by solving an appropriate system of linear equations:

$L = (x^2-1)\left(\frac{d}{dx}\right)^2 + x\frac{d}{dx} - n^2 = \left((\alpha - x)\frac{d}{dx} + n\right)\cdot L_1 = \left((\alpha - x)\frac{d}{dx} - n\right)\cdot L_2$

It follows that both $L[\alpha^n] = 0$ and $L[\alpha^{-n}] = 0$, hence $L[T_n(x)] = 0$.

Proof 2:

Starting from the above equation, we compute a linear relation for the partial derivatives of F. Write $\partial_x = \frac{d}{dx}$ and $\Delta_z = z\frac{d}{dz}$. One computes

$$\left(1 - 2xz + z^2\right) \cdot F = 1 - xz$$

$$\left(1 - 2xz + z^2\right)^2 \cdot \partial_x F = z - z^3$$

$$\left(1 - 2xz + z^2\right)^3 \cdot \partial_x^2 F = 4z^2 - 4z^4$$

$$\left(1 - 2xz + z^2\right)^2 \cdot \Delta_z F = xz - 2z^2 + xz^3$$

$$\left(1 - 2xz + z^2\right)^3 \cdot \partial_x \Delta_z F = z + 2xz^2 - 6z^3 + 2xz^4 + z^5$$

$$\left(1 - 2xz + z^2\right)^3 \cdot \Delta_z^2 F = xz + (2x^2 - 4)z^2 - (2x^2 - 4)z^4 - xz^5$$

Solve a 6×6 system of linear equations over $Q(x)$ to get

$$x \cdot \partial_x F + (x^2 - 1) \cdot \partial_x^2 F - \Delta_z^2 F = 0$$

This is equivalent to the claimed equation $(x^2 - 1) \cdot T_n''(x) + x \cdot T_n'(x) - n^2 \cdot T_n(x) = 0$.

References

1 Bruno Haible: D-finite power series in several variables. *Diploma thesis, University of Karlsruhe, June 1989.* Sections 2.12 and 2.15.