The Legendre polynomials $P_n(x)$ are defined through

$$P_n(x) = \frac{1}{2^n n!} \cdot \left(\frac{d}{dx}\right)^n (x^2-1)^n$$

(For a motivation of the 2^n in the denominator, look at $P_n(x)$ modulo an odd prime p, and observe that $P_n(x) \equiv P_{p-1-n}(x) \mod p$ for $0 \leq n \leq p-1$. This wouldn't hold if the 2^n factor in the denominator weren't present.)

Theorem:

$P_n(x)$ satisfies the recurrence relation

$$P_0(x) = 1$$

$$(n+1) \cdot P_{n+1}(x) = (2n+1)x \cdot P_n(x) - n \cdot P_{n-1}(x)$$

for $n \geq 0$ and the differential equation $(1-x^2) \cdot P_n''(x) - 2x \cdot P_n'(x) + (n^2 + n) \cdot P_n(x) = 0$ for all $n \geq 0$.

Proof:

Let $F := \sum_{n=0}^{\infty} P_n(x) \cdot z^n$ be the generating function of the sequence of polynomials. It is the diagonal series of the power series

$$G := \sum_{m,n=0}^{\infty} \frac{1}{2^n m!} \cdot \left(\frac{d}{dx}\right)^m (x^2-1)^n \cdot y^m \cdot z^n$$

Because the Taylor series development theorem holds in formal power series rings (see [1], section 2.16), we can simplify

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \left(\sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(\frac{d}{dx}\right)^m (x^2-1)^n \cdot y^m \right) \cdot z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \left((x+y)^2 - 1\right)^n \cdot z^n \\ &= \frac{1}{1 - \frac{1}{2}\left((x+y)^2-1\right) z} \end{aligned}$$

We take over the terminology from the "diag_rational" paper; here $R = Q[x]$ and $M = Q[[x]]$ (or, if you like it better, $M = H(C)$, the algebra of functions holomorphic in the entire complex plane). $G \in M[[y,z]]$ is rational; hence

F is algebraic over $R[z]$. Let's proceed exactly as in the "diag_series" paper. $F(z^2)$ is the coefficient of t^0 in

$$G(zt, \frac{z}{t}) = \frac{2t}{2t - ((x+zt)^2 - 1)\, z} = \frac{2t}{-z^3 \cdot t^2 + 2(1 - xz^2) \cdot t + (z - x^2 z)}$$

The splitting field of the denominator is $L = Q(x)(z)(\alpha)$ where

$$\alpha_{1/2} = \frac{1 - xz^2 \pm \sqrt{1 - 2xz^2 + z^4}}{z^3}$$

$$\alpha = \alpha_1 = \frac{2}{z^3} - \frac{2x}{z} + \frac{1 - x^2}{2} z + \cdots \in Q(x)[[z]][\frac{1}{z}]$$

$$\alpha_2 = \frac{x^2 - 1}{2} z + \cdots \in Q(x)[[z]][\frac{1}{z}]$$

The partial fraction decomposition of $G(zt, \frac{z}{t})$ reads

$$G(zt, \frac{z}{t}) = -\frac{2}{z^3} \cdot \frac{1}{\alpha_1 - \alpha_2} \cdot \left(\frac{\alpha_1}{t - \alpha_1} - \frac{\alpha_2}{t - \alpha_2} \right)$$

It follows that

$$F(z^2) = -\frac{2}{z^3} \cdot \frac{1}{\alpha_1 - \alpha_2} \cdot \left(\frac{\alpha_1}{0 - \alpha_1} - 0 \right) = \frac{1}{\sqrt{1 - 2xz^2 + z^4}}$$

hence

$$F(z) = \frac{1}{\sqrt{1 - 2xz + z^2}}$$

It follows that $(1 - 2xz + z^2) \cdot \frac{d}{dz} F + (z - x) \cdot F = 0$. This is equivalent to the claimed recurrence.

Starting from the closed form for F, we compute a linear relation for the partial derivatives of F. Write $\partial_x = \frac{d}{dx}$ and $\Delta_z = z \frac{d}{dz}$. One computes

$$F = 1 \cdot F$$

$$\left(1 - 2xz + z^2\right) \cdot \partial_x F = z \cdot F$$

$$\left(1 - 2xz + z^2\right)^2 \cdot \partial_x^2 F = 3z^2 \cdot F$$

$$\left(1 - 2xz + z^2\right) \cdot \Delta_z F = (xz - z^2) \cdot F$$

$$\left(1 - 2xz + z^2\right)^2 \cdot \partial_x \Delta_z F = (z + xz^2 - 2z^3) \cdot F$$

$$\left(1 - 2xz + z^2\right)^2 \cdot \Delta_z^2 F = \left(xz + (x^2 - 2)z^2 - xz^3 + z^4\right) \cdot F$$

Solve a homogeneous 5×6 system of linear equations over $Q(x)$ to get

$$\left(1 - 2xz + z^2\right)^2 \cdot \left((-2x) \cdot \partial_x F + (1 - x^2) \cdot \partial_x^2 F + \Delta_z F + \Delta_z^2 F\right) = 0$$

Divide by the first factor to get

$$(-2x) \cdot \partial_x F + (1 - x^2) \cdot \partial_x^2 F + \Delta_z F + \Delta_z^2 F = 0$$

This is equivalent to the claimed equation $(1-x^2) \cdot P_n''(x) - 2x \cdot P_n'(x) + (n^2 + n) \cdot P_n(x) = 0$.

References

1 Bruno Haible: D-finite power series in several variables. *Diploma thesis, University of Karlsruhe, June 1989*. Sections 2.14, 2.15 and 2.22.