The Hermite polynomials $H_n(x)$ are defined through

$$H_n(x) = (-1)^n e^{x^2} \cdot \left(\frac{d}{dx}\right)^n \left(e^{-x^2}\right)$$

Theorem:

$H_n(x)$ satisfies the recurrence relation

$$H_0(x) = 1$$

$$H_{n+1}(x) = 2x \cdot H_n(x) - 2n \cdot H_{n-1}(x)$$

for $n \geq 0$ and the differential equation $H_n''(x) - 2x \cdot H_n'(x) + 2n \cdot H_n(x) = 0$ for all $n \geq 0$.

Proof:

Let $F := \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$ be the exponential generating function of the sequence of polynomials. Then, because the Taylor series development theorem holds in formal power series rings (see [1], section 2.16), we can simplify

$$\begin{aligned} F &= e^{x^2} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d}{dx}\right)^n \left(e^{-x^2}\right) \cdot (-z)^n \\ &= e^{x^2} \cdot e^{-(x-z)^2} \\ &= e^{2xz - z^2} \end{aligned}$$

It follows that $\frac{d}{dz} F = (2x - 2z) \cdot F$. This is equivalent to the claimed recurrence.

Starting from this equation, we compute a linear relation for the partial derivatives of F. Write $\partial_x = \frac{d}{dx}$ and $\Delta_z = z\frac{d}{dz}$. One computes

$$F = 1 \cdot F$$

$$\partial_x F = 2z \cdot F$$

$$\partial_x^2 F = 4z^2 \cdot F$$

$$\Delta_z F = (2xz - 2z^2) \cdot F$$

$$\partial_x \Delta_z F = (2z + 4xz^2 - 4z^3) \cdot F$$

$$\Delta_z^2 F = \left(2x \cdot z + (4x^2 - 4) \cdot z^2 - 8x \cdot z^3 + 4 \cdot z^4\right) \cdot F$$

Solve a homogeneous 5×6 system of linear equations over $Q(x)$ to get

$$(-2x) \cdot \partial_x F + \partial_x^2 F + 2 \cdot \Delta_z F = 0$$

This is equivalent to the claimed equation $H_n''(x) - 2x \cdot H_n'(x) + 2n \cdot H_n(x) = 0$.

References

1 Bruno Haible: D-finite power series in several variables. *Diploma thesis, University of Karlsruhe, June 1989*. Sections 2.15 and 2.22.