**Theorem:**

Let $M$ be a torsion-free $R$-module, and $d > 0$. Let

$$f = \sum_{n_1,\ldots,n_d} a_{n_1,\ldots,n_d}\, x_1^{n_1} \cdots x_d^{n_d} \in M[[x_1, \ldots x_d]]$$

be a rational function, i.e. there are $P \in M[x_1, \ldots, x_d]$ and $Q \in R[x_1, \ldots, x_d]$ with $Q(0, \ldots, 0) = 1$ and $Q \cdot f = P$. Then the full diagonal of $f$

$$g = \sum_{n=0}^{\infty} a_{n,\ldots,n}\, x_1^n$$

is a D-finite element of $M[[x_1]]$, w.r.t. $R[x_1]$ and $\{\partial_{x_1}\}$.

**Proof:**

From the hypotheses, $M[[x_1, \ldots, x_d]]$ is a torsion-free differential module over $R[x_1, \ldots, x_d]$ w.r.t. the derivatives $\{\partial_{x_1}, \ldots, \partial_{x_d}\}$, and $f$ is a D-finite element of $M[[x_1, \ldots, x_d]]$ over $R[x_1, \ldots, x_d]$ w.r.t. $\{\partial_{x_1}, \ldots, \partial_{x_d}\}$. Now apply the general diagonal theorem ([1], section 2.18) $d - 1$ times.

**Theorem:**

Let $R$ be an integral domain of characteristic 0 and $M$ simultaneously a torsion-free $R$-module and a commutative $R$-algebra without zero divisors. Let

$$f = \sum_{m,n \geq 0} a_{m,n} x^m y^n \in M[[x, y]]$$

be a rational function. Then the diagonal of $f$

$$g = \sum_{n=0}^{\infty} a_{n,n}\, x^n$$

is algebraic over $R[x]$.

**Motivation of proof:**

The usual proof ([2]) uses complex analysis and works only for $R = M = C$. The idea is to compute

$$g(x^2) = \frac{1}{2\pi i} \oint_{|z|=1} f(xz, \frac{x}{z}) \frac{dz}{z}$$

This integral, whose integrand is a rational function in $x$ and $z$, is calculated using the residue theorem. Since $f(x,y)$ is continuous at $(0,0)$, there is a $\delta > 0$ such that $f(x,y) \neq \infty$ for $|x| < \delta$, $|y| < \delta$. It follows that for all $\varepsilon > 0$ and $|x| < \delta\varepsilon$ all the poles of $f(xz, \frac{x}{z})$ are contained in $\{z : |z| < \varepsilon\} \cup \{z : |z| > \frac{1}{\varepsilon}\}$. Thus the poles of $f(xz, \frac{x}{z})$, all algebraic functions of $x$ – let's call them $\zeta_1(x), \ldots \zeta_s(x)$ –, can be divided up into those for which $|\zeta_i(x)| = O(|x|)$ as $x \to 0$ and those for which $\frac{1}{|\zeta_i(x)|} = O(|x|)$ as $x \to 0$. It follows from the residue theorem that for $|x| < \delta$

$$g(x^2) = \sum_{\zeta = 0 \vee \zeta = O(|x|)} \operatorname*{Res}_{z=\zeta} f(xz, \frac{x}{z})$$

This is algebraic over $C(x)$. Hence $g(x)$ is algebraic over $C(x^{1/2})$, hence also algebraic over $C(x)$.

**Proof:**

Let

$$h(x,z) := f(xz, \frac{x}{z}) = \sum_{m,n=0}^{\infty} a_{m,n} x^{m+n} z^{m-n} \in M[[xz, xz^{-1}]]$$

Then $g(x^2)$ is the coefficient of $z^0$ in $h(x,z)$. Let $N(x,z) := z^d Q(xz, \frac{x}{z})$ (with $d := \max(\deg_y P, \deg_y Q)$) be "the denominator" of $h(x,z)$. We have $N(x,z) \in R[x,z]$ and $N \neq 0$ (because $N(0,z) = z^d$). Let $K$ be the quotient field of $R$. Thus $N(x,z) \in K[x][z] \setminus \{0\}$.

It is well-known (see [3], p.64, or [4], chap. IV, §2, prop. 8, or [5], chap. III, §1) that the splitting field of $N(x,z)$ over $K(x)$ can be embedded into a field $L((x^{1/r}))$, where $r$ is a positive integer and $L$ is a finite-algebraic extension field of $K$, i.e. a simple algebraic extension $L = K(\alpha) = K\alpha^0 + \cdots + K\alpha^{u-1}$.

$\widetilde{M} := (R \setminus \{0\})^{-1} \cdot M$ is a $K$-vector space and a commutative $K$-algebra without zero divisors. $\widehat{M} := \widetilde{M}\alpha^0 + \cdots + \widetilde{M}\alpha^{u-1}$ is an $L$-vector space and a commutative $L$-algebra without zero divisors.

$$\begin{aligned} \widehat{M}\langle\langle x,z \rangle\rangle &:= \widehat{M}[[x^{1/r} \cdot z, x^{1/r} \cdot z^{-1}, x^{1/r}]][x^{-1/r}] \\ &= \left\{ \sum_{m,n} c_{m,n} x^{m/r} z^n : c_{m,n} \neq 0 \Rightarrow |n| \leq m + O(1) \right\} \end{aligned}$$

is an $L$-algebra which contains $\widehat{M}((x^{1/r}))$.

Since $N(x,z)$ splits into linear factors in $L((x^{1/r}))[z]$, $N(x,z) = l\prod_{i=1}^{s}(z-\zeta_i(x))^{k_i}$, there exists a partial fraction decomposition of $h(x,z) = \frac{P(xz,\frac{x}{z})}{Q(xz,\frac{x}{z})} = \frac{z^d P(xz,\frac{x}{z})}{N(x,z)}$ in $\widehat{M}\langle\langle x,z\rangle\rangle$:

$$h(x,z) = \sum_{j=0}^{l} P_j(x)z^j + \sum_{i=1}^{s}\sum_{k=1}^{k_i} \frac{P_{i,k}(x)}{(z-\zeta_i(x))^k}$$

with $P_j(x), P_{i,k}(x) \in \widehat{M}((x^{1/r}))$.

Recall that we are looking for the coefficient of $z^0$ in $h(x,z)$. We compute it separately for each summand.

If $\zeta_i(x) = ax^{m/r} + \ldots$ with $a \in L\setminus\{0\}$, $m>0$, or $\zeta_i(x)=0$, we have

$$\begin{aligned}\frac{1}{(z-\zeta_i(x))^k} &= \frac{1}{z^k}\cdot\frac{1}{\left(1-\frac{\zeta_i(x)}{z}\right)^k}\\ &= \frac{1}{z^k}\cdot\sum_{j=0}^{\infty}\binom{k-1+j}{k-1}\left(\frac{\zeta_i(x)}{z}\right)^j\\ &= \sum_{j=0}^{\infty}\binom{k-1+j}{k-1}\frac{\zeta_i(x)^j}{z^{k+j}}\end{aligned}$$

hence the coefficient of $z^0$ in $\frac{P_{i,k}(x)}{(z-\zeta_i(x))^k}$ is 0.

If $\zeta_i(x) = ax^{m/r} + \ldots$ with $a \in L\setminus\{0\}$, $m<0$, we have

$$\frac{1}{(z-\zeta_i(x))^k} = \frac{1}{(-\zeta_i(x))^k}\cdot\frac{1}{\left(1-\frac{z}{\zeta_i(x)}\right)^k} = \frac{1}{(-\zeta_i(x))^k}\cdot\sum_{j=0}^{\infty}\binom{k-1+j}{k-1}\left(\frac{z}{\zeta_i(x)}\right)^j$$

hence the coefficient of $z^0$ in $\frac{P_{i,k}(x)}{(z-\zeta_i(x))^k}$ is $\frac{P_{i,k}(x)}{(-\zeta_i(x))^k}$.

The case $\zeta_i(x) = ax^{m/r} + \ldots$ with $a \in L\setminus\{0\}$, $m=0$, cannot occur, because it would imply $0 = N(0,\zeta_i(0)) = N(0,a) = a^d$.

Altogether we have

$$g(x^2) = [z^0]h(x,z) = P_0(x) + \sum_{\frac{1}{\zeta_i(x)}=o(x)}\sum_{k=1}^{k_i}\frac{P_{i,k}(x)}{(-\zeta_i(x))^k} \in \widehat{M}((x^{1/r}))$$

Since all $\zeta_i(x)$(in $L((x^{1/r}))$) and all $P_j(x), P_{i,k}(x)$ (in $\widehat{M}((x^{1/r}))$) are algebraic over $K(x)$, the same holds also for $g(x^2)$. Hence $g(x)$ is algebraic over $K(x^{1/2})$, hence also over $K(x)$. After clearing denominators, we finally conclude that $g(x)$ is algebraic over $R[x]$.

# References

1 Bruno Haible: D-finite power series in several variables. *Diploma thesis, University of Karlsruhe, June 1989.* Sections 2.18 and 2.20.

2 M. L. J. Hautus, D. A. Klarner: The diagonal of a double power series. *Duke Math. J.* **38** (1971), 229-235.

3 C. Chevalley: Introduction to the theory of algebraic functions of one variable. *Mathematical Surveys VI. American Mathematical Society.*

4 Jean-Pierre Serre: Corps locaux. *Hermann. Paris* 1968.

5 Martin Eichler: Introduction to the theory of algebraic numbers and functions. *Academic Press. New York, London* 1966.