Application to Euler's Constant

Put
$$f(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!^2}$$
$$g(x) := \sum_{n=0}^{\infty} \left(\frac{1}{1} + \cdots + \frac{1}{n}\right) \frac{x^n}{n!^2}$$

Then
$$\frac{g(x)}{f(x)} = \frac{1}{2}\log x + \gamma + O\left(e^{-4\sqrt{x}}\right).$$

Choose $x = \left\lceil \frac{N \log 2}{4} \right\rceil^2.$