Questionnaire "Logic and Computability"

Summer Term 2023

Contents

1 Propositional Logic

1.1 Declarative Sentences

1.1.1 Look at the following statements and tick them if they are true.

- ☐ "*Give me the butter.*" is a declarative sentence.
- ☐ Questions are always declarative sentences.
- ☐ Declarative sentences can be true and false at the same time.
- ☐ "*My best friend is staying overnight.*" is a declarative sentence.

1.1.2 Model the following sentences as detailed as possible in propositional logic.

(a) Alice will either take the bike or the tram to get to the concert, not both.
(b) Students will have to take an exam at the end of the semester.
(c) If he is hungry and the fridge is not empty, he cooks for himself.

1.1.3 Model the following sentences as detailed as possible in propositional logic.

(a) If the air temperature is above 30°C, then the water temperature is above 20°C and I am able to go for a swim.
(b) Your kid will be safe if and only if it learns to swim.
(c) What time is it?

1.1.4 Model the following sentences as detailed as possible in propositional logic.

(a) Bob will win the lottery, if and only if he gets all the numbers right.
(b) Mozart was born in Salzburg, not in Innsbruck.
(c) If the year is a leap-year, then February will have 29 days.

1.1.5 Model the following sentences as detailed as possible in propositional logic.

(a) Either Alice and Bob are going together to the conference or neither of them is going.
(b) Today is Friday, if and only if yesterday was Thursday and tomorrow is not Sunday.
(c) A model assigns truth values to variables of a propositional formula.

1.1.6 Model the following sentences as detailed as possible in propositional logic.

(a) If a formula is unsat, it cannot be valid.
(b) It can be proven that there exist an infinite number of primes.
(c) A sentence is called declarative, if and only if it can be assigned a truth value.

1.1.7 Model the following sentences as detailed as possible in propositional logic.

(a) Today it will either be foggy or rainy, but not both.
(b) For any program we can prove that the program halts.
(c) The system will never crash, if it got completely verified at design time and is monitored during runtime.

1.2 Syntax of Propositional Logic

1.2.1 Give the definition of well-formed formulas in propositional logic.

1.2.2 How can you determine whether a string is a *well-formed formula* using a parse tree?

1.2.3 Let p, q and r be a atomic propositions. Tick all statements that are true.

- ☐ "$\neg p \wedge \vee \; q$" is a propositional formula.
- ☐ "$(p \wedge q) \vee (r \to p)$" is a propositional formula.
- ☐ "$\neg p$" is a propositional formula.
- ☐ "$\vee$" is a propositional formula.
- ☐ "p" is a propositional formula.

1.2.4 Determine whether the string $\neg(a \vee \neg\neg b)$ is a well-formed formula using the parse tree. Explain your answer.

1.2.5 Determine whether the string $\neg(a \vee \neg b \neg)$ is a well-formed formula using the parse tree. Explain your answer.

1.2.6 Determine whether the string $(a \vee b) \to (\neg(x \neg))$ is a well-formed formula using the parse tree. Explain your answer.

1.3 Semantics of Propositional Logic

1.3.1 Give the definitions of *syntax* of propositional logic and *semantics* of propositional logic. What is the difference between syntax and semantic?

1.3.2 Give the definition of the *semantics* of propositional logic.

1.3.3 Consider a formula φ in propositional logic. Let the number of propositional variables in φ be n. How many rows does the truth table for φ have?

1.3.4 Consider a truth table for a propositional formula φ that has R rows. How many propositional variables does φ have?

1.3.5 Why are truth tables, in general, not used to determine equivalence of large formulas?

1.3.6 Give the definition of a model $\mathcal{M}$ of a formula in propositional logic? Explain the difference between a *full* and a *partial* model.

1.3.7 What is the difference between a *satisfying model* and a *falsifying model* of a formula in propositional logic? Give a satisfying and a falsifying model for the formula $\varphi = a \to b$.

1.3.8 Consider the propositional formula $\varphi = (p \wedge q) \to (q \vee \neg r)$. Fill out the truth table for φ and its subformulas.

p	q	r	$p \wedge q$	$\neg r$	$q \vee \neg r$	$\varphi = (p \wedge q) \to (q \vee \neg r)$
F	**F**	**F**				
F	**F**	**T**				
F	**T**	**F**				
F	**T**	**T**				
T	**F**	**F**				
T	**F**	**T**				
T	**T**	**F**				
T	**T**	**T**				

1.3.9 Consider the propositional formula $\varphi = \neg(\neg p \vee q) \to (p \wedge \neg r)$.

Find a propositional formula ψ that is syntactically different from φ, but semantically equivalent to φ. Show the semantic equivalence of φ and ψ using truth tables.

1.3.10 Given is a formula $\varphi = (p \vee (\neg q \to r)) \wedge (\neg r \to p)$ and a model $\mathcal{M} = \{p = F,\ q = T,\ r = T\}$. Determine the truth value of φ for the given model $\mathcal{M}$ using its parse tree.

1.3.11 Given is a formula $\varphi = ((p \wedge \neg q) \to (p \vee \neg r)) \wedge (\neg q \to \neg r)$ and a model $\mathcal{M} = \{p = T,\ q = F,\ r = T\}$. Determine the truth value of φ for the given model $\mathcal{M}$ using its parse tree.

1.3.12 Given is a formula $\varphi = ((q \to \neg p) \vee r) \to (q \wedge (r \to p))$. Determine a satisfying model $\mathcal{M}_1$ and a falsifying model $\mathcal{M}_2$ using its parse tree.

1.3.13 Given is a formula $\varphi = (\neg(r \leftrightarrow q) \to \neg r) \wedge (\neg(r \to q) \vee (p \to q))$. Determine a satisfying model $\mathcal{M}_1$ and a falsifying model $\mathcal{M}_2$ using its parse tree.

1.4 Semantic Entailment, Equivalence, Satisfiability and Validity

1.4.1 Give the definition of semantic entailment of two propositional formulas.

1.4.2 Give the definition of semantic equivalence of two propositional formulas.

1.4.3 Give the definition of validity of a propositional formula.

1.4.4 Give the definition of satisfiability and unsatisfiability of a propositional formula.

1.4.5 Consider a formula φ in propositional logic. In the following list, tick all statements that are true.

- ☐ If φ is a tautology, a falsifying model can be found.
- ☐ If φ is equivalent to ψ, a satisfying model for φ always satisfies ψ.
- ☐ If φ has no satisfying model, it is called a tautology.
- ☐ If φ semantically entails ψ, a satisfying model for ψ always satisfies φ.

1.4.6 Consider a formula φ in propositional logic. In the following list, tick all statements that are true.

- ☐ If φ is not satisfiable, $\neg\varphi$ is valid.
- ☐ If φ is valid, $\neg\varphi$ is not valid.
- ☐ If φ is valid, $\neg\varphi$ is not satisfiable.
- ☐ If φ is not valid, $\neg\varphi$ is satisfiable.

1.4.7 Consider a formula φ in propositional logic. You want to test whether φ is *valid*. However, you only have a procedure for checking satisfiability. Describe how to use this procedure to determine whether φ is valid.

1.4.8 Given are the truth tables for the propositional logic formulas φ and ψ. Determine whether it holds that $\varphi \models \psi$, $\psi \models \varphi$, or neither.

p	q	r	φ	ψ
F	**F**	**F**	F	F
F	**F**	**T**	T	T
F	**T**	**F**	F	F
F	**T**	**T**	T	T
T	**F**	**F**	F	F
T	**F**	**T**	F	T
T	**T**	**F**	T	T
T	**T**	**T**	T	T

1.4.9 Consider the propositional formulas $\varphi = (p \to q) \vee \neg r$ and $\psi = (\neg r \wedge p) \vee (\neg q \to \neg r)$.

(a) Fill out the truth table for φ and ψ and their subformulas.

p	q	r	$\neg q$	$\neg r$	$p \to q$	$\neg r \wedge p$	$\neg q \to \neg r$	φ	ψ
F	**F**	**F**							
F	**F**	**T**							
F	**T**	**F**							
F	**T**	**T**							
T	**F**	**F**							
T	**F**	**T**							
T	**T**	**F**							
T	**T**	**T**							

(b) Which of the formulas is satisfiable?

(c) Which of the formulas is valid?

(d) Which of the two formulas φ and ψ entails the other?

1.4.10 Consider the propositional formulas $\varphi = (p \vee q) \to r$, and $\psi = r \vee (\neg p \wedge \neg q)$.

(a) Fill out the truth table for φ and ψ (and their subformulas).

p	q	r	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	φ	ψ
F	**F**	**F**						
F	**F**	**T**						
F	**T**	**F**						
F	**T**	**T**						
T	**F**	**F**						
T	**F**	**T**						
T	**T**	**F**						
T	**T**	**T**						

(b) Which of the formulas is satisfiable?

(c) Which of the formulas is valid?

(d) Is φ equivalent to ψ?

(e) Does φ semantically entail ψ?

(f) Does ψ semantically entail φ?

1.4.11 Consider the propositional formula $\varphi = p \to (q \to r)$.

(a) Fill out the truth table for φ and its subformulas.

p	q	r	$(q \to r)$	$\varphi = p \to (q \to r)$
F	**F**	**F**		
F	**F**	**T**		
F	**T**	**F**		
F	**T**	**T**		
T	**F**	**F**		
T	**F**	**T**		
T	**T**	**F**		
T	**T**	**T**		

(b) Is φ satisfiable?

(c) Give a formula ψ that is semantically equivalent to φ, but does not use the "$\to$" connective.

(d) How can you check whether ψ is semantically equivalent to φ?

1.4.12 Consider the propositional formula $\varphi = (p \to q) \land (q \to r) \land (\neg r \lor p)$.

(a) Fill out the truth table for φ (and its subformulas).

p	q	r	$(p \to q)$	$(q \to r)$	$\neg r$	$(\neg r \lor p)$	φ
F	**F**	**F**					
F	**F**	**T**					
F	**T**	**F**					
F	**T**	**T**					
T	**F**	**F**					
T	**F**	**T**					
T	**T**	**F**					
T	**T**	**T**					

(b) Is φ satisfiable?

(c) Is φ valid?

(d) Give a formula ψ that semantically entails φ (i.e., it should be the case that $\psi \models \varphi$).

(e) How can you check, using a truth table, whether ψ semantically entails φ?

1.4.13 Consider the propositional formula $\varphi = (\neg p \to r) \land (r \to \neg p) \land q$.

(a) Fill out the truth table for φ (and its subformulas).

p	q	r	$\neg p$	$(\neg p \to r)$	$(r \to \neg p)$	φ
F	**F**	**F**				
F	**F**	**T**				
F	**T**	**F**				
F	**T**	**T**				
T	**F**	**F**				
T	**F**	**T**				
T	**T**	**F**				
T	**T**	**T**				

(b) Is the negation of φ satisfiable?

(c) Is the negation of φ valid?

(d) Give a formula ψ that semantically entails φ (i.e., it should be the case that $\psi \models \varphi$).

(e) Give a formula ψ such that φ semantically entails ψ (i.e., it should be the case that $\varphi \models \psi$).

1.5 Modelling Example

1.5.1 Use propositional logic to solve Sudoku. Rules: A Sudoku grid consists of a 9x9 square, which is partitioned into nine 3x3 squares. The goal of the game is to write one number from 1 to 9 in each cell in such a way, that each row, each column, and each 3x3-square contains each number exactly once. Usually several numbers are already given.

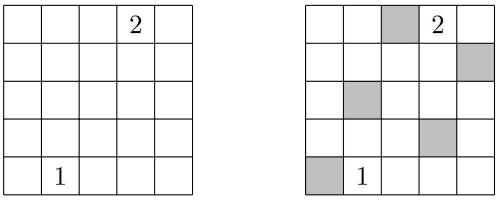
	6		7			1	5	
		3	9			8		
		2	3				4	9
		7			4			
	4			9			8	
			1			4		
6	7				9	3		
		9			2	5		
	2	8			7		6	

Sudoku

In order to model SUDOKU using propositional logic, we first need to define the propositional variables that we want to use in our formula. We define variables x_{ijk} for every row i, for every column j, and for every value k. This encoding yields to 729 variables ranging from x_{111} to x_{999}. Using this variables, define the constraints for the rows, the columns, the 3x3-squares and the predefined numbers.

1.5.2 Describe the Latin Square Puzzle using propositional logic.

In the Latin Square Puzzle one has to color cells in an (n×n) grid such that there is exactly one colored cell in each row and each column. Furthermore, colored cells must not be adjacent to each other (also not diagonally). Numbers contained in certain cells of the grid indicate the exact number of colored cells that have to be adjacent (including diagonally) to it. Numbered cells can contain the numbers 0, 1, 2 and cannot be colored.



Example Latin Square Puzzle and its solution

Find propositional formulas which describe the puzzle and which could be used to solve it. Focus on explaining the concept of the formulas. You do not have to explicitly list all formulas and you do not have to solve the puzzle.

Hints: Use propositional atoms $c_{i,j}$, $c_{i,j,0}$, $c_{i,j,1}$,$c_{i,j,2}$ to represent each cell of the $(n \times n)$ game board. If $c_{i,j}$ has the value *True*, the cell i, j is colored, otherwise it is not colored. If $c_{i,j,x}$ has the value *True*, the cell i, j contains the number x.

Express the following constraints:

(a) There is exactly one colored cell in row i.
(b) No colored cells are adjacent to each other.
(c) No numbered cells can be colored.
(d) Numbered cells are adjacent to the indicated amount of colored cells.