Questionnaire "Logic and Computability"

Summer Term 2023

Contents

7 Predicate Logic

7.1 Predicates and Quantifiers

7.1.1 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Some students like Alice.
(b) Every teacher likes Bob.
(c) Some students like every teacher.
(d) Some students and Bob play a game.
(e) Not every student plays games.
(f) Some teachers play no games.

Solution

$\mathcal{A}$ = People

(a) $\exists x\big(S(x) \land L(x, Alice)\big)$
$S(x) \ldots x$ is a student
$L(x,y) \ldots x$ likes y

(b) $\forall x\big(T(x) \rightarrow L(x, Bob)\big)$
$T(x) \ldots x$ is a teacher
$L(x,y) \ldots x$ likes y

(c) $\exists x\big(S(x) \land \forall y\big(T(y) \rightarrow L(x,y)\big)\big)$
$S(x) \ldots x$ is a student
$T(x) \ldots x$ is a teacher
$L(x,y) \ldots x$ likes y

(d) $P(Bob) \land \exists x\big(S(x) \land P(x)\big)$
$S(x) \ldots x$ is a student
$P(x) \ldots x$ plays a game

(e) $\neg\forall x\big(S(x) \rightarrow P(x)\big)$
$S(x) \ldots x$ is a student
$P(x) \ldots x$ plays a game

(f) $\exists x\big(T(x) \rightarrow \neg P(x)\big)$
$T(x) \ldots x$ is a teacher
$P(x) \ldots x$ plays a game

7.1.2 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Alice has no sister.

(b) A person who wears a crown is either a king or a queen.
(c) Not everybody likes everybody.
(d) Everybody loves somebody.

Solution

$\mathcal{A}$ = People

(a) $\forall x\big(A(x) \to \neg S(x)\big)$
$A(x) \ldots x$ is Alice
$S(x) \ldots x$ has a sister

(b) $\forall x\big(C(x) \to K(x) \lor Q(x)\big)$
$C(x) \ldots x$ wears a crown
$K(x) \ldots x$ is a king
$Q(x) \ldots x$ is a queen

(c) $\neg\forall x \forall y\big(L(x,y)\big)$
$L(x,y) \ldots x$ likes y

(d) $\forall x \exists y\big(L(x,y)\big)$
$L(x,y) \ldots x$ loves y

7.1.3 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) The construction takes a long time, is noisy, and blocks the sun.
(b) If there is no school, at least one parent of each kid has to take vacation and cannot got to work.
(c) All students have to take the exam eventually.

Solution

(a) $x \wedge y \wedge z$
$x \ldots$ the construction side takes a long time
$y \ldots$ the construction side is noisy
$z \ldots$ the construction side blocks the sun

(b) $\neg a \to \forall x \exists y \big(K(x) \wedge P(x,y) \to V(y) \wedge \neg W(y)\big)$
$a \ldots$ there is school
$K(x) \ldots x$ is a kid
$P(x,y) \ldots y$ is parent of x
$V(x) \ldots x$ takes vacation
$W(x) \ldots x$ goes to work
$\mathcal{A} = \textit{people}$

(c) $\forall x \big(S(x) \to E(x)\big)$
$S(x) \ldots x$ is a student
$E(x) \ldots x$ takes the exam
$\mathcal{A} =$ People

7.1.4 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) If all kids wear gloves, then all parents will be happy.
(b) All kids love pizza and spaghetti.
(c) All kids are fun, energetic, and cannot sit still.

Solution

$\mathcal{A} =$ People

(a) $\forall x \big(K(x) \wedge G(x) \to \forall y \big(P(y) \wedge H(y)\big)\big)$
$K(x) \ldots x$ is a kid
$G(x) \ldots x$ wears gloves
$P(x) \ldots x$ is a parent
$H(x) \ldots x$ is happy

(b) $\forall x \big(K(x) \to P(x) \wedge S(x)\big)$
$K(x) \ldots x$ is a kid
$P(x) \ldots x$ loves pizza
$S(x) \ldots x$ loves spaghetti

(c) $\forall x \big(K(x) \to F(x) \wedge E(x) \wedge \neg S(x)\big)$
$K(x) \ldots x$ is a kid
$F(x) \ldots x$ is fun
$E(x) \ldots x$ is energetic
$S(x) \ldots x$ can sit still

7.1.5 Consider the following declarative sentence (known as *Goldbach's Conjecture*):

"Every even integer greater than 2 is equal to the sum of two prime numbers."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Solution

There is no solution available for this question yet.

7.1.6 Consider the following declarative sentences:

"Every person who has the same parents as John Doe and is different from John Doe himself is a sibling of John Doe."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.
Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

Solution

There is no solution available for this question yet.

7.1.7 Translate the following sentences into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

(a) Nobody knows everybody.
(b) All birds can fly, except for penguins and ostrichs.
(c) Not all birds can fly, but some birds can fly.
(d) All kids are cute and quite if and only if they are sleeping

Solution

There is no solution available for this question yet.

7.1.8 Translate the following sentence into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

Every even integer greater than 2 is equal to the sum of two prime numbers.

Solution

There is no solution available for this question yet.

7.1.9 Consider the following declarative sentence:

"For every natural number it holds that it is prime if and only if there is no smaller natural number, except for 1, that divides it."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

Solution

There is no solution available for this question yet.

7.1.10 *"For all triangles it holds it is a scalene triangle iff all its sides have different lengths and all its angles have different measure."*

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Solution

There is no solution available for this question yet.

7.1.11 *"Everyone gets a break once in a while, but the break cannot last forever"*

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

Solution

There is no solution available for this question yet.

7.1.12 Model the following sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Every integer is greater or equal to one.

(b) For any two integers, their sum is smaller than their product

Solution

There is no solution available for this question yet.

7.2 Syntax of Predicate Logic

7.2.1 The syntax of predicate logic is defined via 2 types of sorts: *terms* and *formulas*. What are terms and what are formulas? Give examples for both.

Solution

terms: Terms talk about objects, they are elements of the domain: individual objects like Alice or Bob, variables since they represent objects like x, y, function symbols since they refer to objects like $m(x)$ or $x + y$

formulas: Formulas have a truth value. Each predicate is a formula, e.g. $S(x)$, $P(x)$, $\forall x\big(S(x) \to P(x)\big)$

7.2.2 Give the definition of the syntax of predicate logic.

Solution

- $\mathcal{V}$: Defines the set of variable symbols, e.g., x, y, z.
- $\mathcal{F}$: Defines the set of function symbols, e.g., f, g, h.
- $\mathcal{P}$: Defines the set of predicate symbols, e.g., P, Q, R.

Terms are defined as follows:

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, \ldots t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots t_n)$ is a term.
- Nothing else is a term.

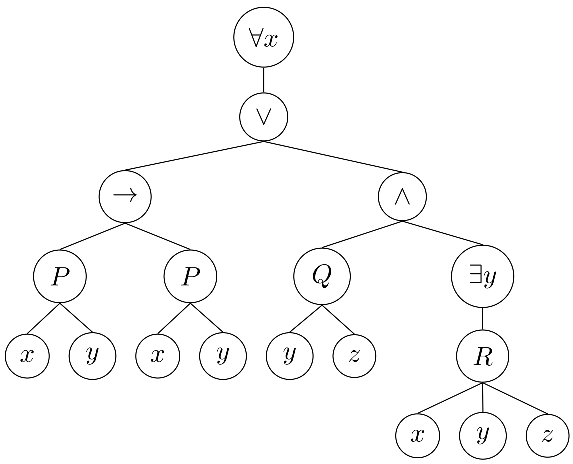
Formulas are defined as follows:

- If $P \in \mathcal{P}$ is a predicate with arity $n > 0$ and $t_1, t_2, \ldots t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots t_n)$ is a formula.
- If φ is a formula, then $\neg\varphi$ is a formula.
- If φ and ψ are formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ are formulas.
- If φ is a formula and x is a variable, then $(\forall x \varphi)$ and $(\exists x \varphi)$ are formulas.
- Nothing else is a formula.

7.2.3 Draw the syntax tree for the following formula:

$$\forall x \left(\big(P(x,y) \rightarrow P(x,x)\big) \vee \big(Q(y,z) \wedge \exists y\, R(x,y,z)\big) \right)$$

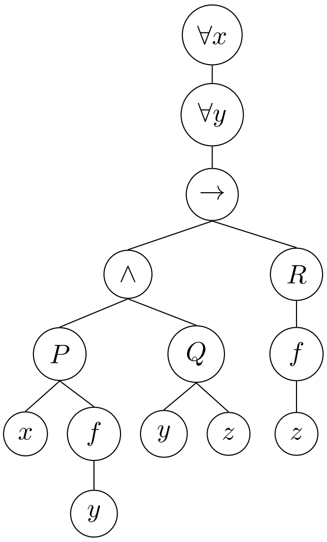
Solution



7.2.4 Draw the syntax tree for the following formula:

$$\forall x \exists y\ (P(x, f(y)) \wedge Q(y,z) \rightarrow R(f(z))).$$

Solution



7.2.5 Given is the following formula in predicate logic

$$\varphi = \forall x \exists y \Big(\big(Q(x,y) \wedge P(x,y) \big) \rightarrow \big(R(y,x) \wedge P(x,y) \big) \Big).$$

Draw the syntax tree for φ.

Solution

There is no solution available for this question yet.

7.2.6 Given is the following formula in predicate logic

$$\varphi = \exists x \forall y \Big(\big(P(x,y) \rightarrow Q(x,y) \big) \vee \big(P(y,x) \rightarrow R(x,y) \big) \Big).$$

Draw the syntax tree for φ.

Solution

There is no solution available for this question yet.

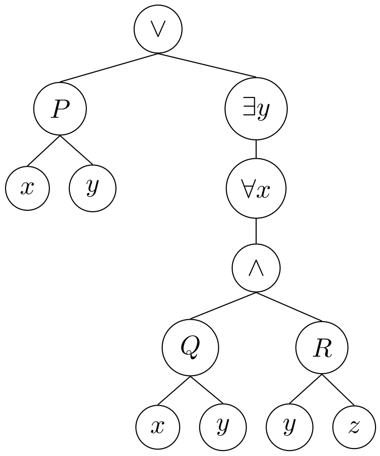
7.3 Free and Bound Variables

7.3.1 Given the formula

$$P(x,y) \vee \exists y \forall x \ \big(Q(x,y) \wedge R(y,z) \big),$$

construct a syntax tree for φ and determine the *scope* of its quantifiers and which occurrences of the variables are *free* and which are *bound*.

Solution



Free variables: x, y, z
Bound variables: x, y

7.3.2 In the context of predicate logic:

(a) What is the scope of a quantifier?
(b) What is the difference between *free* and *bound* variables?

Given an example that shows the difference.

Solution

There is no solution available for this question yet.

7.3.3 In the context of predicate logic, give a definition of *substitution* of variables.

Solution

There is no solution available for this question yet.

7.3.4 What does it mean to *substitute a term t for a variable x in a predicate logic formula?* Which rules to you have to consider when performing substitution? Give an example.

Solution

There is no solution available for this question yet.

7.3.5 Consider the following formula.

$$\varphi := \forall y\big(P(x) \wedge Q(y)\big) \vee \big(R(y) \wedge Q(x)\big)$$

(a) Compute $\varphi[f(x)/x]$.
(b) Compute $\varphi[f(y)/x]$.
(c) Compute $\varphi[f(z)/x]$.

Solution

There is no solution available for this question yet.

7.3.6 Consider the following formula.

$$\varphi := \forall y\big(P(x) \wedge Q(y)\big) \to \exists x\big(R(y) \wedge Q(x)\big)$$

(a) Compute $\varphi[f(y)/x]$.
(b) Compute $\varphi[f(x)/y]$.
(c) Compute $\varphi[k/z]$.
(d) Compute $\varphi[x/z]$.

Solution

There is no solution available for this question yet.

7.3.7 Given the formula

$$\varphi = \forall x \exists z \; \big(\neg P(x) \vee Q(y, f(z))\big) \to \big(\neg \exists x \; P(y) \wedge Q(f(x), z)\big).$$

(a) Compute $\varphi[f(y)/x]$.
(b) Compute $\varphi[f(x)/y]$.
(c) Compute $\varphi[k/z]$.
(d) Compute $\varphi[x/z]$.

Solution

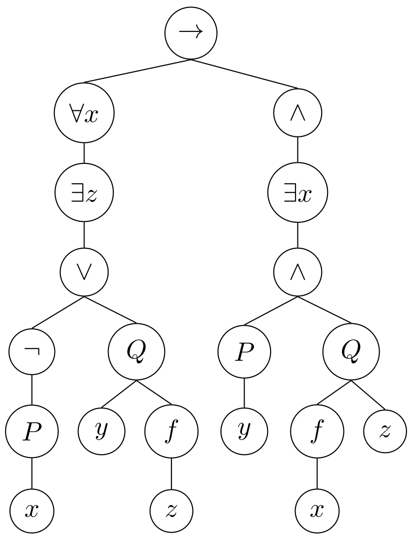
There is no solution available for this question yet.

7.3.8 Given the formula

$$\varphi = \forall x \exists z \; \big(\neg P(x) \vee Q(y, f(z))\big) \to \big(\exists x \; P(y) \wedge Q(f(x), z)\big),$$

construct a syntax tree for φ and determine the *scope* of its quantifiers and which occurrences of the variables are *free* and which are *bound*.

Solution



Free variables: y, z
Bound variables: x, z

7.4 Semantics of Predicate Logic

7.4.1 Give a model $\mathcal{M}$ for the following formula:

$$\varphi \coloneqq \exists x \forall y P(x, y).$$

Solution

$\mathcal{M} : \mathcal{A} = \{a, b\}$
$P^{\mathcal{M}} = \{(a, a), (a, b)\}$

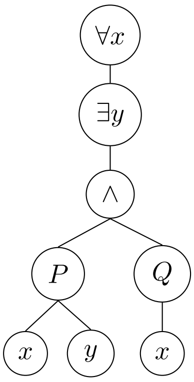
7.4.2 Consider the formula

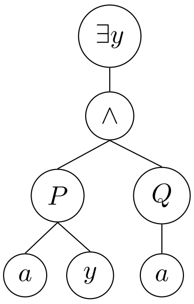
$$\varphi \coloneqq \forall x \exists y (P(x, y) \land Q(x)).$$

Give a model the satisfies the formula and a second one that falsifies the formula.

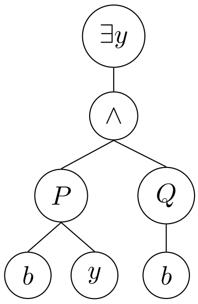
Show using the parse tree why your models satisfy are falsify the formula.

Solution

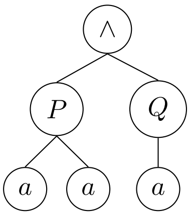




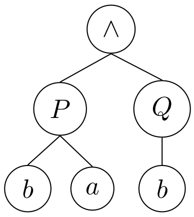
Subtree: $x = a$



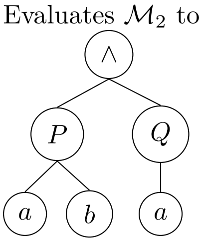
Subtree: $x = b$



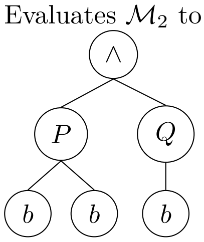
Subtree: $x = a \wedge y = a$
Evaluates $\mathcal{M}_1$ to true.
Evaluates $\mathcal{M}_2$ to false.



Subtree: $x = b \wedge y = a$
Evaluates $\mathcal{M}_1$ to true.
Evaluates $\mathcal{M}_2$ to false.



Subtree: $x = a \wedge y = b$
Evaluates $\mathcal{M}_1$ to true.
Evaluates $\mathcal{M}_2$ to false.



Subtree: $x = b \wedge y = b$
Evaluates $\mathcal{M}_1$ to true.
Evaluates $\mathcal{M}_2$ to false.

$\mathcal{M}_1 : \mathcal{A} = \{a, b\}$
$P =$ true
$Q =$ true
$\mathcal{M}_1 \models \varphi$

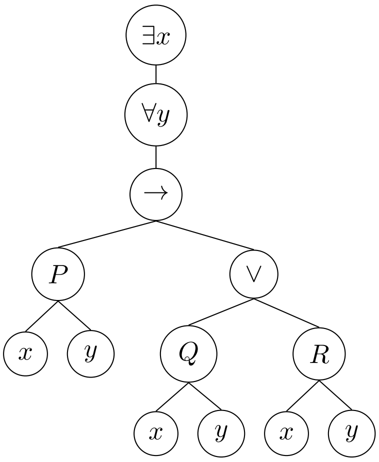
$\mathcal{M}_2 : \mathcal{A} = \{a, b\}$
$P =$ true
$Q =$ false
$\mathcal{M}_2 \not\models \varphi$

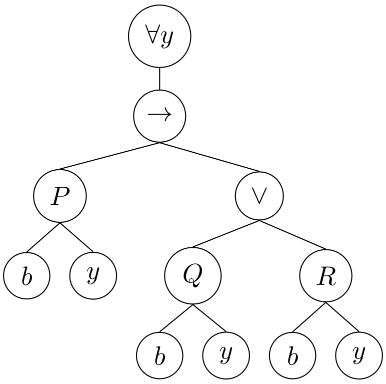
7.4.3 Consider the formula

$$\varphi = \exists x \forall y \left(P(x,y) \rightarrow (Q(x,y) \vee R(x,y)) \right).$$

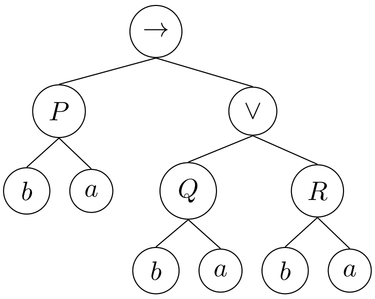
Does the following model $\mathcal{M}$ satisfy the formula?
$\mathcal{A} = \{a, b\}$
$P^{\mathcal{M}} = \{(a,a),(a,b)\}$
$Q^{\mathcal{M}} = \{(a,a),(b,a)\}$
$R^{\mathcal{M}} = \{(a,a),(b,b)\}$

Solution

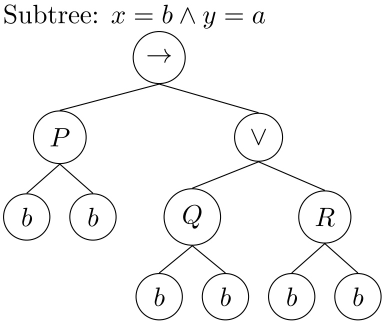




Subtree: $x = b$



Subtree: $x = b \wedge y = a$



Subtree: $x = b \wedge y = b$

The model $\mathcal{M}$ satisfies the formula.

7.4.4 Give the definition of a model in predicate logic. Discuss what needs to be defined in a *model* of a predicate logic formula. Give an example for each data that could be contained in a model.

Solution

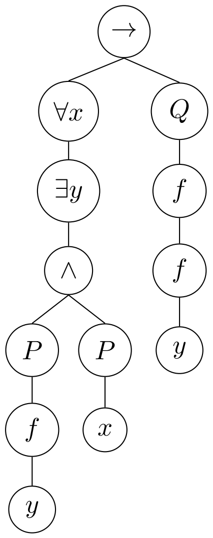
Definition - Model in Predicate Logic. *A model $\mathcal{M}$ consists of the following set of data:*

- *A non-empty set $\mathcal{A}$, the universe/domain of concrete values;*
- *for each nullary function symbol $f \in \mathcal{F}$, a concrete element $f^{\mathcal{M}} \in \mathcal{A}$;*
- *for each nullary predicate symbol $P \in \mathcal{P}$, a truth value;*
- *for each function symbol $f \in \mathcal{F}$ with arity $n > 0$ a concrete function $f^{\mathcal{M}} : \mathcal{A}^n \to \mathcal{A}$;*
- *for each predicate smybol $P \in \mathcal{P}$ with arity $n > 0$: subset $P^{\mathcal{M}} \subseteq \mathcal{A}^n$;*
- *for any free variable* var: *a lookup-table $t : \mathrm{var} \to \mathcal{A}$.*

7.4.5 For the following formula in *Predicate Logic*, find a *model* that satisfies the formula and one that does not. Draw a syntax tree and state all free variables while solving this task.

(a) $\forall x \exists y \left(P(f(y)) \ \land \ P(x)\right) \ \to \ Q(f(f(y)))$

Solution



Free variables: y

$\mathcal{M}_1 : \mathcal{A} = \{a, b\}$
$P^{\mathcal{M}_1} =$ true
$Q^{\mathcal{M}_1} =$ true
$\mathcal{M}_1 \models \varphi$
f can be defined arbitrarily in this case.
$\mathcal{M}_2 : \mathcal{A} = \{a, b\}$
$P^{\mathcal{M}_2} =$ true
$Q^{\mathcal{M}_2} =$ false
$\mathcal{M}_2 \not\models \varphi$ f can be defined arbitrarily in this case.

7.4.6 In the following list, tick all items that are required for a complete model of a formula φ in predicate logic.

- ☐ A non-empty, possibly infinite set of values for variables and functions.
- ☐ A concrete value for every bound variable in φ.
- ☐ A concrete value for free bound variable in φ.
- ☐ A definition for each predicate in φ, detailing for which values/tuples the predicate returns *true*.
- ☐ A definition for each function in φ, detailing for which values/tuples the predicate returns *true*.

Solution

There is no solution available for this question yet.

7.4.7 Given is the following formula in predicate logic

$$\varphi = \forall x \exists y \Big(\big(Q(x, y) \land P(x, y)\big) \rightarrow \big(R(y, x) \land P(x, y)\big) \Big)$$

and the model $\mathcal{M}$:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}} = \{(m, a) | m \in \mathcal{A}\}$
- $Q^{\mathcal{M}} = \{(b, m) | m \in \mathcal{A}\}$
- $R^{\mathcal{M}} = \{(a, b), (b, a), (b, b)\}$

Does the model $\mathcal{M}$ satisfy the formula φ? Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

Solution

There is no solution available for this question yet.

7.4.8 For the formula below, find one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

$$\exists x(P(x) \wedge Q(f(x))) \vee (\neg P(x) \wedge \neg Q(f(x))$$

Solution

There is no solution available for this question yet.

7.4.9 For the formula below, find one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

$$\neg \forall x((P(x) \rightarrow P(y)) \wedge P(x))$$

Solution

There is no solution available for this question yet.

7.4.10 For the formula below, state one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a syntax tree and evaluate your models with the help of this syntax tree.

$$\forall x \exists y(P(f(x), y) \wedge \neg P(x, f(y)))$$

Solution

There is no solution available for this question yet.

7.4.11 For each of the formulas in predicate logic below, find a model that satisfies the formula and one that does not. Draw a syntax tree and state all free variables while solving this task.

(a) $\neg \forall x((P(x) \rightarrow P(y)) \wedge P(x))$
(b) $\forall x \exists y(P(x, y) \wedge \neg P(f(x), f(y)))$

Solution

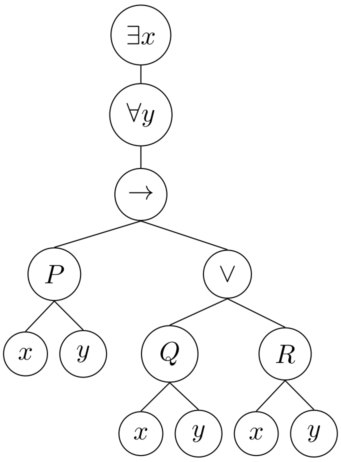
There is no solution available for this question yet.

7.4.12 Consider the sentence $\varphi = \exists x \forall y (P(x, y) \rightarrow (Q(x, y) \vee R(x, y)))$. Does the following model satisfy φ?

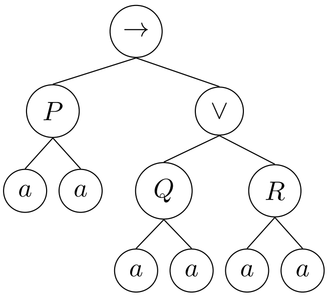
The model M consists of:

- $A = \{a, b, c\}$
- $P^M = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b)\}$
- $Q^M = \{(a, m) | m \in A\}$
- $R^M = \{(a, a), (b, a), (a, c), (b, c), (c, c)\}$

Solution

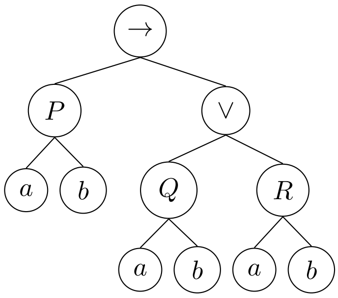


$x = a \wedge y = a$



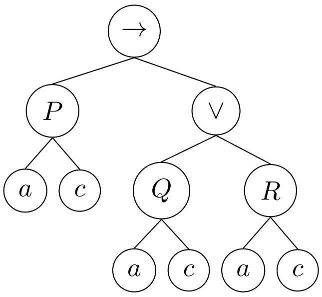
$P(a,a) \to (Q(a,a) \vee R(a,a)) =$
$\top \to (\top \vee \top) = \top$

$x = a \wedge y = b$



$P(a,b) \to (Q(a,b) \vee R(a,b)) =$
$\top \to (\top \vee \bot) = \top$

$x = a \wedge y = c$



$P(a,c) \to (Q(a,c) \vee R(a,c)) =$
$\bot \to (\top \vee \top) = \top$

We have found an x, such that for all y the formula evaluates to *true*. Therefore

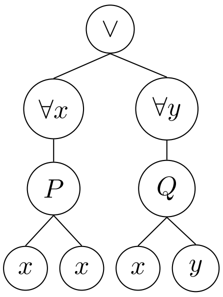
$$M \models \varphi.$$

7.4.13 For each of the formulas of predicate logic below, find a model that satisfies the formula and one that does not. Draw a syntax tree and state all free variables.

(a) $\forall x(P(x,x)) \vee \forall y(Q(x,y))$

(b) $\neg\forall x((Q(f(x)) \rightarrow P(f(f(x)))) \wedge \neg Q(x))$

Solution



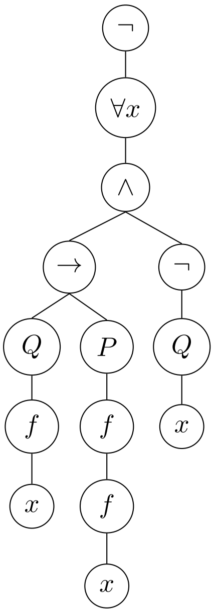
The x in the right subformula is a free variable.

An example for a *satisfying* model $\mathcal{M}_{SAT}$:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}_{SAT}} = \{(a, a), (b, b)\}$
- $Q^{\mathcal{M}_{SAT}} = \{(a, a), (a, b)\}$

An example for an *unsatisfying* model $\mathcal{M}_{UNSAT}$:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}_{UNSAT}} = \{(a, a)\}$
- $Q^{\mathcal{M}_{UNSAT}} = \{(a, a)\}$



An example for a *satisfying* model $\mathcal{M}_{SAT}$:

- $\mathcal{A} = \{a, b\}$
- $f^{\mathcal{M}_{SAT}} = \{x \rightarrow x\}$
- $P^{\mathcal{M}_{SAT}} = \top$
- $Q^{\mathcal{M}_{SAT}} = \bot$

An example for an *unsatisfying* model $\mathcal{M}_{UNSAT}$:

- $\mathcal{A} = \{a, b\}$
- $f^{\mathcal{M}_{UNSAT}} = \{x \rightarrow x\}$
- $P^{\mathcal{M}_{UNSAT}} = \top$
- $Q^{\mathcal{M}_{UNSAT}} = \bot$