Questionnaire “Logic and Computability”

Summer Term 2023

Contents

4 Natural Deduction for Propositional Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion. For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

4.1 Rules for Natural Deduction

4.1.1 Give the definition of a sequent. Give an example of a sequent and name the parts the sequent consists of.

Solution

A sequent is an expression of the form

$$\varphi_1, \varphi_2, \ldots, \varphi_n \;\vdash\; \psi.$$

$\varphi_1, \varphi_2, \ldots, \varphi_n$ are called premises. ψ is called the conclusion. The premises entail the conclusion. This means that for any valid sequence, we can proof that the conclusion follows from the premises.

4.1.2 Look at the following statements and tick them if they are true.

- ☒ In a sequent, premises entail a conclusion.
- ☐ In a sequent, conclusions entail a premise.
- ☐ A sequent is valid, independent on whether a proof can be found.
- ☒ A sequent is valid, if a proof for it can be found.

4.1.3 Explain the concept of boxes in deduction rules and why they are needed. What does it mean if you make an *assumption* within a box? Where is this assumption valid?

Solution

Assumptions assume that within the box, a certain formula holds that can be used to prove something within the box. The assumption is only valid within the box. Therefore, any formulas proven within the box are only valid inside the box, because they are proven under a given assumption that is only valid in the scope of the box.

4.1.4 Explain the *OR-elimination* ($\vee$-*e*) rule of the natural deduction calculus. In particular, why does it rule require two boxes?

Solution

From a given formula $\varphi \lor \psi$, we want to proof some other formula χ. We only know that φ *or* ψ holds. It could be that both of them are true, but it could also be that only ψ is true, or only φ is true. Sine we don't know which sub-formula is true, we have to give two separate proofs:

- First box: We assume φ is true and need to find a proof for χ.
- Second box: We assume ψ is true and need to find a proof for χ.

Only if we can prove χ in the first and in the second box, then we can conclude that χ holds also outside of the box.
The $\lor_e$ rules says that we can only derive χ from $\varphi \lor \psi$ if we can derive χ from the assumption φ as well as from the assumption ψ. Formally the rule is written as:

$$\dfrac{\varphi \lor \psi \qquad \boxed{\begin{array}{l}\varphi \text{ ass.}\\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{l}\psi \text{ ass.}\\ \vdots \\ \chi\end{array}}}{\chi}\lor_e$$

4.1.5 Why are there two rules for the $\lor$-*introduction* rule. Explain, why you are able to connect any formula to a certain formula φ using the connective $\lor$.

Solution

$$\dfrac{\varphi}{\varphi \lor \psi}\lor_i 1 \qquad \dfrac{\varphi}{\psi \lor \varphi}\lor_i 2$$

If we know that φ holds, we can derive that $\varphi \lor \psi$ holds and that $\psi \lor \varphi$ holds. This is true for any ψ.

4.1.6 $\neg\neg\neg p \land q, \neg\neg r \ \vdash \ r \land \neg p \land \neg\neg q$

Solution

This sequent is provable.

1.	$\neg\neg\neg p \land q$	premise
2.	$\neg\neg r$	premise
3.	$\neg\neg\neg p$	$\land_e 11$
4.	q	$\land_e 21$
5.	r	$\neg\neg_e 2$
6.	$\neg p$	$\neg\neg_e 3$
7.	$\neg\neg q$	$\neg\neg_i 4$
8.	$r \land \neg p$	$\land_i 5, 6$
9.	$r \land \neg p \land \neg\neg q$	$\land_i 8, 7$

4.1.7 $p, p \to q, p \to (q \to r) \ \vdash \ r$

Solution

1.	p	premise
2.	$p \to q$	premise
3.	$p \to (q \to r)$	premise
4.	$q \to r$	$\to_e$ 1, 3
5.	q	$\to_e$ 1, 2
6.	r	$\to_e$ 4, 5

4.1.8 $\neg p \to q, \neg\neg\neg q \wedge r \ \vdash \ p \wedge \neg\neg\neg q$

Solution

This sequent is provable.

1.	$\neg p \to q$	premise
2.	$\neg\neg\neg q \wedge r$	premise
3.	$\neg\neg\neg q$	$\wedge_e 12$
4.	$\neg q$	$\neg\neg_e 3$
5.	$\neg\neg p$	MT 1, 4
6.	p	$\neg\neg_e 5$
7.	$p \wedge \neg\neg\neg q$	$\wedge_i 6, 3$

4.1.9 $\neg p \to (q \to r), \neg p, \neg r \ \vdash \ \neg q$

Solution

1.	$\neg p \to (q \to r)$	premise
2.	$\neg p$	premise
3.	$\neg r$	premise
4.	$q \to r$	$\to_e$ 1, 2
5.	$\neg q$	MT 4, 3

4.1.10 $p \to q, \ q \to r \ \vdash \ p \to r$

Solution

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	p	assumption
4.	q	$\to_e$ 3, 1
5.	r	$\to_e$ 4, 2
6.	$p \to r$	$\to_i$ 3 − 5

4.1.11 $p \to (q \wedge r), q \to s \ \vdash \ p \to (s \wedge r)$

Solution

This sequent is provable.

1.	$p \to (q \land r)$	premise
2.	$q \to s$	premise
3.	p	assumption
4.	$q \land r$	$\to_e$ 1, 3
5.	q	$\land_e 14$
6.	s	$\to_e$ 2, 5
7.	r	$\land_e 24$
8.	$s \land r$	$\land_i 6, 7$
9.	$p \to (s \land r)$	$\to_i$ 3 − 8

4.1.12 $p \land q, r \to s \;\vdash\; (p \lor (r \to s)) \land (q \lor ((t \lor r) \to u))$

Solution

This sequent is provable.

1.	$p \land q$	premise
2.	$r \to s$	premise
3.	p	$\land_e 11$
4.	$p \lor (r \to s)$	$\lor_i 13$
5.	q	$\land_e 21$
6.	$q \lor ((t \lor r) \to u)$	$\lor_i 15$
7.	$(p \lor (r \to s)) \land (q \lor ((t \lor r) \to u))$	$\land_i 4, 6$

4.1.13 $q \to r \;\vdash\; (p \lor q) \to (p \lor r)$

Solution

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	$\lor_i 12$
5.	q	assumption
6.	r	$\to_e$ 5, 1
7.	$p \lor r$	$\lor_i 26$
8.	$p \lor r$	$\lor_e 2, 3 - 4, 5 - 7$
9.	$p \lor q \to (p \lor r)$	$\to_i$ 2 − 8

4.1.14 $p \to \neg q, q \;\vdash\; \neg p$

Solution

1.	$p \to \neg q$	premise
2.	q	premise
3.	p	assumption
4.	$\neg q$	$\to_e$ 3, 1
5.	$\bot$	$\neg_e 2, 4$
6.	$\neg p$	$\neg_i 3 - 5$

4.1.15 Solve this task without using the Modus Tollens.

$$\neg p \to \neg q, q \ \vdash \ p$$

Solution

1.	$\neg p \to \neg q$	premise
2.	q	premise
3.	$\neg p$	assumption
4.	$\neg q$	$\to_e$ 3, 1
5.	$\bot$	$\neg_e 2, 4$
6.	p	PBC $3 - 5$

4.1.16 $\vdash \ ((p \to q) \to p) \to p$

Solution

This sequent is provable.

1.	$(p \to q) \to p$	assumption
2.	$\neg p$	assumption
3.	$\neg(p \to q)$	MT 1, 2
4.	p	assumption
5.	$\bot$	$\neg_e 2, 4$
6.	q	$\bot_e 5$
7.	$p \to q$	$\to_i$ $4 - 6$
8.	$\bot$	$\neg_e 3, 7$
9.	p	PBC $2 - 8$
10.	$((p \to q) \to p) \to p$	$\to_i$ $1 - 9$

4.1.17 $\neg q \lor \neg p \ \vdash \ \neg(q \land p)$

Solution

This sequent is provable.

1.	$\neg q \vee \neg p$	premise
2.	$q \wedge p$	assumption
3.	$\neg q$	assumption
4.	q	$\wedge_e 12$
5.	$\bot$	$\neg_e 3, 4$
6.	$\neg p$	assumption
7.	p	$\wedge_e 22$
8.	$\bot$	$\neg_e 6, 7$
9.	$\bot$	$\vee_e 1, 3-5, 6-8$
10.	$\neg(q \wedge p)$	$\neg_i 2-9$

4.1.18 $p \vee \neg\neg q, \neg p \wedge \neg q \ \vdash \ s \vee \neg t$

Solution

This sequent is provable.

1.	$p \vee \neg\neg q$	premise
2.	$\neg p \wedge \neg q$	premise
3.	p	assumption
4.	$\neg p$	$\wedge_e 12$
5.	$\bot$	$\neg_e 3, 4$
6.	$s \vee \neg t$	$\bot_e 5$
7.	$\neg\neg q$	assumption
8.	$\neg q$	$\wedge_e 22$
9.	$\bot$	$\neg_e 7, 8$
10.	$s \vee \neg t$	$\bot_e 9$
11.	$s \vee \neg t$	$\vee_e 1, 3-6, 7-10$

4.1.19 $p \to q \ \vdash \ \neg p \vee q$

Solution

1.	$p \to q$	premise
2.	$\neg p \vee p$	LEM
3.	$\neg p$	assumption
4.	$\neg p \vee q$	$\vee_i 13$
5.	p	assumption
6.	q	→e 1,5
7.	$\neg p \vee q$	$\vee_i 26$
8.	$\neg p \vee q$	$\vee_e 2, 3-5, 5-7$

4.1.20 $\vdash \ \neg(p \wedge q) \vee p$

Solution

This sequent is provable.

1.	$p \vee \neg p$	LEM
2.	p	assumption
3.	$\neg(p \wedge q) \vee p$	$\vee_i 2 2$
4.	$\neg p$	assumption
5.	$p \wedge q$	assumption
6.	p	$\wedge_e 1 5$
7.	$\bot$	$\neg_e 6, 4$
8.	$\neg(p \wedge q)$	$\neg_i 5-7$
9.	$\neg(p \wedge q) \vee p$	$\vee_i 1 8$
10.	$\neg(p \wedge q) \vee p$	$\vee_e 1, 2-3, 4-9$

4.1.21 $\neg\neg k \to (l \vee m), \neg\neg\neg l \to m \;\vdash\; \neg k \vee (l \vee \neg\neg m)$

Solution

This sequent is provable.

1.	$\neg\neg k \to (l \wedge m)$	premise
2.	$\neg\neg\neg l \to m$	premise
3.	$m \vee \neg m$	LEM
4.	m	assumption
5.	$\neg\neg m$	$\neg\neg_i 4$
6.	$l \vee \neg\neg m$	$\vee_i 2 5$
7.	$\neg k \vee (l \vee \neg\neg m)$	$\vee_i 2 6$
8.	$\neg m$	assumption
9.	$\neg\neg\neg\neg l$	MT $2, 8$
10.	$\neg\neg l$	$\neg\neg_e 9$
11.	l	$\neg\neg_e 10$
12.	$l \vee \neg\neg m$	$\vee_i 1 11$
13.	$\neg k \vee (l \vee \neg\neg m)$	$\vee_i 2 12$
14.	$\neg k \vee (l \vee \neg\neg m)$	$\vee_e 3, 4-7, 8-13$

4.1.22 $\neg(a \wedge b) \vee \neg c \;\vdash\; \neg(a \wedge b) \to c \vee a$

Solution

This sequent is not provable.
$\mathcal{M} : a = F, b = F, c = F$
$\mathcal{M} \models \neg(a \wedge b) \vee \neg c$
$\mathcal{M} \not\models \neg(a \wedge b) \to c \vee a$

4.1.23 $(s \vee \neg u) \to t \;\vdash\; (\neg s \wedge u) \vee t$

Solution

This sequent is provable.

1.	$(s \vee \neg u) \rightarrow t$	premise
2.	$s \vee \neg s$	LEM
3.	s	assumption
4.	$s \vee \neg u$	$\vee_i 13$
5.	t	$\rightarrow_e 4, 1$
6.	$(\neg s \wedge u) \vee t$	$\vee_i 25$
7.	$\neg s$	assumption
8.	$u \vee \neg u$	LEM
9.	u	assumption
10.	$\neg s \wedge u$	$\wedge_i 7, 9$
11.	$(\neg s \wedge u) \vee t$	$\vee_i 110$
12.	$\neg u$	assumption
13.	$s \vee \neg u$	$\vee_i 212$
14.	t	$\rightarrow_e 131$
15.	$(\neg s \wedge u) \vee t$	$\vee_i 214$
16.	$(\neg s \wedge u) \vee t$	$\vee_e 8, 9 - 11, 12 - 15$
17.	$(\neg s \wedge u) \vee t$	$\vee_e 2, 3 - 6, 7 - 16$

4.1.24 Provide a natural deduction proof for the following sequent without using the *Modus Tollens* rule:

$$\varphi \rightarrow \psi, \neg\psi \;\vdash\; \neg\varphi$$

Solution

1.	$\neg\psi$	premise
2.	$\varphi \rightarrow \psi$	premise
3.	φ	assumption
4.	ψ	$\rightarrow_e 1, 2$
5.	$\perp$	$\neg_e 1, 4$
6.	$\neg\varphi$	$\neg_i 3 - 5$

4.1.25 $\neg\neg p \wedge \neg\neg q, r \wedge s \;\vdash\; (p \wedge r) \wedge \neg\neg s$

Solution

1.	$\neg\neg p \wedge \neg\neg q$	premise
2.	$r \wedge s$	premise
3.	$\neg\neg p$	$\wedge_e 11$
4.	p	$\neg\neg_e 3$
5.	r	$\wedge_e 12$
6.	s	$\wedge_e 22$
7.	$\neg\neg s$	$\neg\neg_i 6$
8.	$p \wedge r$	$\wedge_i 4, 5$
9.	$(p \wedge r) \wedge \neg\neg s$	$\wedge_i 8, 7$

4.1.26 $(\neg p \to q) \wedge (q \to r), \neg r \ \vdash \ \neg\neg\neg r \wedge \neg p$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = T, q = F, r = F$
$\mathcal{M} \models (\neg p \to q) \wedge (q \to r), \neg r$
$\mathcal{M} \not\models \neg\neg\neg r \wedge \neg p$

4.1.27 $(p \to q) \to r \ \vdash \ \neg r \wedge \neg s \to \neg(p \to q)$

Solution

1.	$(p \to q) \to r$	premise
2.	$\neg r \wedge \neg s$	assumption
3.	$\neg r$	$\wedge_e 12$
4.	$\neg(p \to q)$	MT $1, 3$
5.	$\neg r \wedge \neg s \to \neg(p \to q)$	$\to_i 2 - 4$

4.1.28 $p \to q \ \vdash \ (r \to p) \to (r \to q)$

Solution

1.	$p \to q$	premise
2.	$r \to p$	assumption
3.	r	assumption
4.	p	$\to_e 2, 3$
5.	q	$\to_e 1, 4$
6.	$r \to q$	$\to_i 3 - 5$
7.	$(r \to p) \to (r \to q)$	$\to_i 2 - 6$

4.1.29 $p \to q, p \wedge (r \vee q) \ \vdash \ (q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$

Solution

1.	$p \to q$	premise
2.	$p \wedge (r \vee q)$	premise
3.	$q \to p$	assumption
4.	$r \vee q$	$\wedge_e 22$
5.	p	$\wedge_e 12$
6.	q	$\to_e 1, 5$
7.	$(s \wedge t) \vee q$	$\vee_i 26$
8.	$((s \wedge t) \vee q) \wedge (r \vee q)$	$\wedge_i 7, 4$
9.	$(q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$	$\to_i 3 - 7$

4.1.30 $p \vee q, \neg p \vee r \ \vdash \ q \vee r$

Solution

1.	$p \vee q$	premise
2.	$\neg p \vee r$	premise
3.	p	assumption
4.	$\neg p$	assumption
5.	$\bot$	$\neg_e 3, 4$
6.	$q \vee r$	$\bot_e 5$
7.	r	assumption
8.	$q \vee r$	$\vee_i 27$
9.	$q \vee r$	$\vee_e 2, 4 - 6, 7 - 8$
10.	q	assumption
11.	$q \vee r$	$\vee_i 110$
12.	$q \vee r$	$\vee_e 1, 3 - 9, 10 - 11$

4.1.31 $p \to q, p \wedge r \vee q \ \vdash \ (q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$

Solution

1.	$p \to q$	premise
2.	$(p \land r) \lor q$	premise
3.	$q \to p$	assumption
4.	$p \land r$	assumption
5.	r	$\land_e 24$
6.	$r \lor q$	$\lor_i 15$
7.	q	assumption
8.	$r \lor q$	$\lor_i 27$
9.	$r \lor q$	$\lor_e 2, 4-6, 7-8$
10.	$p \land r$	assumption
11.	p	$\land_e 110$
12.	q	$\to_e 1, 11$
13.	$(s \land t) \lor q$	$\lor_i 212$
14.	q	assumption
15.	$(s \land t) \lor q$	$\lor_i 214$
16.	$(s \land t) \lor q$	$\lor_e 2, 10-13, 14-15$
17.	$((s \land t) \lor q) \land (r \lor q)$	$\land_i 16, 9$
18.	$(q \to p) \to ((s \land t) \lor q) \land (r \lor q)$	$\to_i 3-17$

4.1.32 $p \lor q, p \to r, \neg s \to \neg q \;\vdash\; r \lor s$

Solution

1.	$p \lor q$	premise
2.	$p \to r$	premise
3.	$\neg s \to \neg q$	premise
4.	p	assumption
5.	r	$\to_e 4, 2$
6.	$r \lor s$	$\lor_i 15$
7.	q	assumption
8.	s	MT $3, 7$
9.	$r \lor s$	$\lor_i 28$
10.	$r \lor s$	$\lor_e 1, 4-6, 7-9$

4.1.33 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

If I press the button, the window opens.
The window is not open.
Therefore, I didn't press the button.

Solution

b = I press the button.
w = The window is open.

$b \to w, \neg w \ \vdash \ \neg b$

1.	$b \to w$	premise
2.	$\neg w$	premise
3.	b	assumption
4.	w	$\to_e$ 1, 3
5.	$\bot$	$\neg_e 2, 4$
6.	$\neg b$	$\neg_i 3 - 5$

4.1.34 $\neg q \vee p \ \vdash \ q \to (p \vee r)$

Solution

1.	$\neg q \vee p$	premise
2.	q	assumption
3.	$\neg q$	assumption
4.	$\bot$	$\neg_e 2, 3$
5.	$p \vee r$	$\bot_e 4$
6.	p	assumption
7.	$p \vee r$	$\vee_i 16$
8.	$p \vee r$	$\vee_e 1, 3 - 5, 6 - 7$
9.	$q \to (p \vee r)$	$\to_i$ 2 − 8

4.1.35 $p \to (q \vee r), \neg q \wedge \neg r \ \vdash \ \neg p$

Solution

1.	$p \to (q \vee r)$	premise
2.	$\neg q \wedge \neg r$	premise
3.	p	assumption
4.	$q \vee r$	$\to_e$ 1, 3
5.	q	assumption
6.	$\neg q$	$\wedge_e 12$
7.	$\bot$	$\neg_e 5, 6$
8.	r	assumption
9.	$\neg r$	$\wedge_e 22$
10.	$\bot$	$\neg_e 8, 9$
11.	$\bot$	$\vee_e 4, 5 - 7, 8 - 10$
12.	$\neg p$	$\neg_i 3 - 11$

4.1.36 $\neg(q \vee p) \ \vdash \ \neg q \wedge p$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = F, q = F$
$\mathcal{M} \models \neg(q \vee p)$
$\mathcal{M} \not\models \neg q \wedge p$

4.1.37 $\vdash \ (p \to q) \vee (q \to r)$

Solution

1.	$q \vee \neg q$	LEM
2.	q	assumption
3.	p	assumption
4.	q	copy 2
5.	$p \to q$	$\to_i$ 3 − 4
6.	$(p \to q) \vee (q \to r)$	$\vee_i 15$
7.	$\neg q$	assumption
8.	q	assumption
9.	$\bot$	$\neg_e 7, 8$
10.	r	$\bot_e 9$
11.	$q \to r$	$\to_i$ 8 − 10
12.	$(p \to q) \vee (q \to r)$	$\vee_i 211$
13.	$(p \to q) \vee (q \to r)$	$\vee_e 1, 2-6, 7-12$

4.1.38 $(p \to q) \wedge (q \to p) \ \vdash \ (p \wedge q) \vee (\neg p \wedge \neg q)$

Solution

1.	$(p \to q) \wedge (q \to p)$	premise
2.	$p \vee \neg p$	LEM
3.	p	assumption
4.	$p \to q$	$\wedge_e 11$
5.	q	$\to_e$ 3, 4
6.	$p \wedge q$	$\wedge_i 3, 5$
7.	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$\vee_i 16$
8.	$\neg p$	assumption
9.	$q \to p$	$\wedge_e 21$
10.	$\neg q$	MT 8, 9
11.	$\neg p \wedge \neg q$	$\wedge_i 8, 10$
12.	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$\vee_i 211$
13.	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$\vee_e 2, 3-7, 8-12$

4.1.39 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

> If I press the button, the window opens.
> I pressed the button.

Therefore, the window is open.

Solution

Translation:
p : I press the button.
q : The window is open.

If I press the button, the window opens.	$p \to q$
I pressed the button.	p
The window is open.	q

Sequent: $p \to q, p \vdash q$

This sequent is provable.

1.	$p \to q$	premise
2.	p	premise
3.	q	$\to_e$ 1, 2

4.1.40 $(p \to (q \to r)) \wedge (q \to (r \to s)) \vdash p \to (q \to s)$

Solution

This sequent is provable.

1.	$(p \to (q \to r)) \wedge (q \to (r \to s))$	premise
2.	p	assumption
3.	$p \to (q \to r)$	$\wedge_e 1 1$
4.	$q \to (r \to s)$	$\wedge_e 2 1$
5.	$q \to r$	$\to_e$ 2, 3
6.	q	assumption
7.	r	$\to_e$ 5, 6
8.	$r \to s$	$\to_e$ 4, 6
9.	s	$\to_e$ 7, 8
10.	$q \to s$	$\to_i$ 6 − 9
11.	$p \to (q \to s)$	$\to_i$ 2 − 10

4.1.41 $\vdash (p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r))$

Solution

This sequent is provable.

1.	$p \to (q \land r)$	assumption
2.	p	assumption
3.	$q \land r$	$\to_e 1, 2$
4.	q	$\land_e 13$
5.	$p \to q$	$\to_i 2 - 4$
6.	p	assumption
7.	$q \land r$	$\to_e 1, 6$
8.	r	$\land_e 27$
9.	$p \to r$	$\to_i 6 - 8$
10.	$(p \to q) \land (p \to r)$	$\land_i 5, 9$
11.	$(p \to (q \land r)) \to ((p \to q) \land (p \to r))$	$\to_i 1 - 10$

4.1.42 $(p \to (q \to r)) \lor (q \to (r \to s)) \ \vdash \ p \to (q \to s)$

Solution

This sequent is not provable.
$\mathcal{M} : p = T, q = T, r = T, s = F$
$\mathcal{M} \models (p \to (q \to r)) \lor (q \to (r \to s))$
$\mathcal{M} \not\models p \to (q \to s)$

4.1.43 $\vdash \ (p \land q) \to \neg(\neg p \lor \neg q)$

Solution

This sequent is provable.

1.	$p \land q$	assumption
2.	p	$\land_e 11$
3.	q	$\land_e 21$
4.	$\neg p \lor \neg q$	assumption
5.	$\neg p$	assumption
6.	$\bot$	$\neg_e 2, 5$
7.	$\neg q$	assumption
8.	$\bot$	$\neg_e 3, 7$
9.	$\bot$	$\lor_e 4, 5 - 6, 7 - 8$
10.	$\neg(\neg p \lor \neg q)$	$\neg_i 4 - 9$
11.	$(p \land q) \to \neg(\neg p \lor \neg q)$	$\to_i 1 - 10$

4.1.44 $\neg(p \land q) \lor \neg(r \land s) \ \vdash \ (\neg p \land \neg r) \lor (\neg q \land \neg s)$

Solution

This sequent is not provable.
$\mathcal{M} : p = T, q = T, r = T, s = F$
$\mathcal{M} \models \neg(p \wedge q) \vee \neg(r \wedge s)$
$\mathcal{M} \not\models (\neg p \wedge \neg r) \vee (\neg q \wedge \neg s)$

4.1.45 $p \to q \wedge \neg r, r \to s \vee \neg q \;\vdash\; p \to \neg s$

Solution

This sequent is not provable.
$\mathcal{M} : p = T, q = T, r = F, s = T$
$\mathcal{M} \models p \to q \wedge \neg r$
$\mathcal{M} \models r \to s \vee \neg q$
$\mathcal{M} \not\models p \to \neg s$

4.1.46 $p \to q \vee r, \neg q \;\vdash\; p \to r$

Solution

This sequent is provable.

1.	$p \to q \vee r$	premise
2.	$\neg q$	premise
3.	p	assumption
4.	$q \vee r$	$\to_e$ 1, 3
5.	q	assumption
6.	$\bot$	$\neg_e 2, 5$
7.	r	$\bot_e 6$
8.	r	assumption
9.	r	copy 8
10.	r	$\vee_e 4, 5-7, 8-9$
11.	$p \to r$	$\to_i$ 3 − 10

4.1.47 $\neg(p \wedge q) \to r, p \to \neg q \;\vdash\; r$

Solution

This sequent is provable.

1.	$\neg(p \wedge q) \to r$	premise
2.	$p \to \neg q$	premise
3.	$\neg r$	assumption
4.	$\neg\neg(p \wedge q)$	MT 1
5.	$p \wedge q$	$\neg\neg_e 4$
6.	p	$\wedge_e 15$
7.	q	$\wedge_e 26$
8.	$\neg q$	$\to_e 2, 6$
9.	$\bot$	$\bot_e 7, 8$
10.	r	PBC $3 - 9$

4.1.48 $(p \wedge q) \vee (p \wedge r) \ \vdash \ p \wedge (q \vee r)$

Solution

This sequent is provable.

1.	$(p \wedge q) \vee (p \wedge r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge_e 12$
4.	$p \wedge r$	assumption
5.	p	$\wedge_e 14$
6.	p	$\vee_e 1, 2 - 3, 4 - 5$
7.	$p \wedge q$	assumption
8.	q	$\wedge_e 27$
9.	$q \vee r$	$\vee_i 8$
10.	$p \wedge r$	assumption
11.	r	$\wedge_e 210$
12.	$q \vee r$	$\vee_i 11$
13.	$q \vee r$	$\vee_e 1, 7 - 9, 10 - 12$
14.	$p \wedge (q \vee r)$	$\wedge_i 6, 13$

4.1.49 $p \wedge q \to r, p \to \neg q \ \vdash \ \neg r$

Solution

This sequent is not provable.
$\mathcal{M} : p = F, q = Fr = T$
$\mathcal{M} \models p \wedge q \to r$
$\mathcal{M} \models p \to \neg q$
$\mathcal{M} \not\models \neg r$

4.1.50 $\vdash \ (p \to q) \vee \neg q$

Solution

This sequent is provable.

1.	$\neg((p \to q) \vee \neg q)$	assumption
2.	$p \to q$	assumption
3.	$(p \to q) \vee \neg q$	$\vee_i 13$
4.	$\bot$	$\neg_e 3, 1$
5.	$\neg(p \to q)$	$\neg_i 2 - 4$
6.	q	assumption
7.	p	assumption
8.	q	copy 6
9.	$p \to q$	$\to_i 7 - 8$
10.	$\bot$	$\neg_e 9, 5$
11.	$\neg q$	$\neg_i 6 - 10$
12.	$(p \to q) \vee \neg q$	$\vee_i 11$
13.	$\bot$	$\neg_e 12, 1$
14.	$(p \to q) \vee \neg q$	PBC $1 - 13$

4.1.51 $\vdash \ (p \to q) \wedge \neg q$

Solution

This sequent is not provable.
$\mathcal{M} : p = T, q = T$
$\mathcal{M} \not\models (p \to q) \wedge \neg q$

4.1.52 $(p \to q) \wedge (q \to r), \neg r \vee q \ \vdash \ \neg p \vee r$

Solution

This sequent is provable.

1.	$(p \to q) \wedge (q \to r)$	premise
2.	$\neg r \vee q$	premise
3.	$p \to q$	$\wedge_e 1$
4.	$q \to r$	$\wedge_e 1$
5.	$\neg r$	assumption
6.	$\neg q$	MT $4, 5$
7.	$\neg p$	MT $3, 6$
8.	$\neg p \vee r$	$\vee_i 7$
9.	q	assumption
10.	r	$\to_e 9, 4$
11.	$\neg p \vee r$	$\vee_i 10$
12.	$\neg p \vee r$	$\vee_e 2, 5 - 8, 9 - 10$

4.1.53 $(p \wedge q) \to (p \wedge r), q \ \vdash \ \neg p \vee r$

Solution

This sequent is provable.

1.	$(p \wedge q) \rightarrow (p \wedge r)$	premise
2.	q	premise
3.	p	assumption
4.	$p \wedge q$	$\wedge_i 3, 2$
5.	$p \wedge r$	$\rightarrow_e 4, 1$
6.	r	$\wedge_e 26$
7.	$p \rightarrow r$	$\rightarrow_i 3-6$
8.	$\neg(\neg p \vee r)$	assumption
9.	$\neg p$	assumption
10.	$\neg p \vee r$	$\vee_i 9$
11.	$\bot$	$\neg_e 10, 8$
12.	p	PBC $9-11$
13.	r	$\rightarrow_e 12, 7$
14.	$\neg p \vee r$	$\vee_i 213$
15.	$\bot$	$\neg_e 14, 8$
16.	$\neg p \vee r$	PBC $8-15$

4.1.54 $p \rightarrow q, q \rightarrow r, \neg(p \wedge r) \ \vdash \ q \rightarrow p$

Solution

This sequent is not provable.
$\mathcal{M} : p = F, q = T, r = T$
$\mathcal{M} \models p \rightarrow q$
$\mathcal{M} \models q \rightarrow r$
$\mathcal{M} \models \neg(p \wedge r)$
$\mathcal{M} \not\models q \rightarrow p$

4.1.55 $(p \wedge q) \rightarrow (r \vee s), (r \wedge p \wedge q) \ \vdash \ (q \rightarrow r) \wedge (s \rightarrow q)$

Solution

This sequent is provable.

1.	$(p \wedge q) \to (r \vee s)$	premise
2.	$r \wedge p \wedge q$	premise
3.	$p \wedge q$	$\wedge_e 21$
4.	$r \vee s$	$\to_e 1, 3$
5.	q	assumption
6.	r	$\wedge_e 12$
7.	$q \to r$	$\to_i 5 - 6$
8.	s	assumption
9.	q	$\wedge_e 23$
10.	$s \to q$	$\to_i 8 - 9$
11.	$(q \to r) \wedge (s \to q)$	$\wedge_i 7, 10$

4.1.56 $(p \wedge r) \ \vdash \ (q \to p) \wedge (s \to r)$

Solution

1.	$p \wedge r$	premise
2.	p	$\wedge_e 11$
3.	r	$\wedge_e 21$
4.	s	assumption
5.	r	copy 2
6.	$s \to r$	$\to_i 4 - 5$
7.	q	assumption
8.	p	copy 1
9.	$q \to p$	$\to_i 7 - 8$
10.	$(q \to p) \wedge (s \to r)$	$\wedge_i 6, 9$

4.1.57 $p \to (q \vee r), q \to s \ \vdash \ \neg(p \to (q \wedge s))$

Solution

This sequent is not provable.

$\mathcal{M} : p = T, q = T, r = T, s = T$
$\mathcal{M} \models p \to (q \vee r)$
$\mathcal{M} \models q \to s$
$\mathcal{M} \models \neg(p \to (q \wedge s))$

4.1.58 $\neg(p \vee \neg q) \ \vdash \ p$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = F, q = T$
$\mathcal{M} \models \neg(p \lor \neg q)$
$\mathcal{M} \not\models p$

4.1.59 $p \to q \vdash ((p \lor q) \to p) \land (p \to (p \lor q))$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = F, q = T$
$\mathcal{M} \models p \to q$
$\mathcal{M} \not\models ((p \lor q) \to p) \land (p \to (p \lor q))$

4.1.60 $p \lor q, \neg q \lor r \;\vdash\; r$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = T, q = F, r = F$
$\mathcal{M} \models p \lor q, \neg q \lor r$
$\mathcal{M} \not\models r$

4.1.61 $(p \land q) \to (\neg r \land \neg s), \neg r \land \neg s \;\vdash\; p$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = F, q = F, r = F, s = F$
$\mathcal{M} \models (p \land q) \to (\neg r \land \neg s), \neg r \land \neg s$
$\mathcal{M} \not\models p$

4.1.62 (a) If I am ill, I go to the doctor.
I am ill.
Therefore, I go to the doctor.

(b) If I am ill, I go to the doctor.
I go to the doctor.
Therefore, I am ill.

(c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor.
I did not go to the doctor.
Therefore, I am not ill.

Solution

Translation:
p : I am ill.
q : I go to the doctor.

(a) If I am ill, I go to the doctor. $p \to q$
I am ill. p
Therefore, I go to the doctor. $\vdash q$

Sequent: $p \to q, p \vdash q$

1.	$p \to q$	premise
2.	p	premise
3.	q	$\to_e$ 2, 1

(b) If I am ill, I go to the doctor. $p \to q$
I go to the doctor. q
Therefore, I am ill. $\vdash p$

Sequent: $p \to q, q \vdash p$
This sequent is not provable.

$\mathcal{M} : p = F, q = T$
$\mathcal{M} \models p \to q, q$
$\mathcal{M} \not\models p$

(c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor. $p \to q$
I did not go to the doctor. $\neg q$
Therefore, I am not ill. $\neg p$

Sequent: $p \to q, \neg q \vdash \neg p$

1.	$p \to q$	premise
2.	$\neg q$	premise
3.	p	assumption
4.	q	$\to_e$ 3, 1
5.	$\bot$	$\neg_e 4, 2$
6.	$\neg p$	$\neg_i 3-5$

4.1.63 (a) $(p \wedge q) \wedge \neg r \vdash q \vee r$
(b) $(p \vee q) \wedge \neg r \vdash q \wedge r$

Solution

(a) This sequent is provable.

1.	$(p \wedge q) \wedge \neg r$	premise
2.	$p \wedge q$	$\wedge_e 11$
3.	q	$\wedge_e 22$
4.	$q \vee r$	$\vee_i 13$

(b) This sequent is not provable.
$\mathcal{M} : p = T, q = T, r = F$
$\mathcal{M} \models (p \vee q) \wedge \neg r$
$\mathcal{M} \not\models q \wedge r$

4.1.64 $p \wedge q, q \to \neg\neg r \;\vdash\; p \wedge r$

Solution

This sequent is provable.

1.	$p \wedge q$	premise
2.	$q \to \neg\neg r$	premise
3.	q	$\wedge_e 21$
4.	$\neg\neg r$	$\to_e 2, 3$
5.	r	$\neg\neg_e 4$
6.	p	$\wedge_e 11$
7.	$p \wedge r$	$\wedge_i 6, 5$

4.1.65 $(p \wedge q) \;\vdash\; ((\neg(p \wedge q)) \to r) \vee ((q \to s) \wedge t)$

Solution

This sequent is provable.

1.	$p \wedge q$	premise
2.	$\neg(p \wedge q)$	assumption
3.	$p \wedge q$	copy 1
4.	$\bot$	$\neg_e 2, 3$
5.	r	$\bot_e 3$
6.	$\neg(p \wedge q) \to r$	$\to_i 2-5$
7.	$((\neg(p \wedge q)) \to r) \vee ((q \to s) \wedge t)$	$\vee_i 16$

4.1.66 $\neg t, (p \wedge r) \to t \;\vdash\; (r \to s) \wedge (p \to q)$

Solution

This sequent is not provable.
$\mathcal{M} : p = T, q = F, r = F, s = F, t = F$
$\mathcal{M} \models \neg t$
$\mathcal{M} \models (p \wedge r) \to t$
$\mathcal{M} \not\models (r \to s) \wedge (p \to q)$

4.1.67 $p \to q, q \to r \;\vdash\; r$.

Solution

This sequent is not provable.
$\mathcal{M} : p = F, q = F, r = F$
$\mathcal{M} \models p \to q, q \to r$
$\mathcal{M} \not\models r$

4.1.68 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

If I press the button, the window opens.
The window is open.
Therefore, I pressed the button.

Solution

Translation:
p : Press button.
q : Open window.

If I press the button, the window opens.	$p \to q$
The window is open.	q
Therefore, I pressed the button.	$\vdash p$

sequent: $p \to q, q \vdash p$

This sequent is not provable.
$\mathcal{M} : p = F, q = T$
$\mathcal{M} \models p \to q, q$
$\mathcal{M} \not\models p$

4.2 Soundness and Completeness of Natural Deduction

4.2.1 "Natural deduction for propositional logic is *sound* and *complete*." Explain in your own words what this means.

Solution

- *Natural deduction for propositional logic is sound. Therefore, any sequent that can be proven is a correct semantic entailment.*

  Natural deduction is sound. This means that any sequent $\varphi_1, \varphi_2, \ldots \;\vdash\; \psi$ that is provable states a correct semantic entailment $\varphi_1, \varphi_2, \ldots \models \psi$. A correct semantic entailment tells us that under all models that satisfy φ_i for all i the conclusion ψ evaluates to true.

  In short: Anything that is provable by natural deduction is true with respect to semantics.

- *Natural deduction for propositional logic is complete. Therefore, any sequent that is a correct semantic entailment can be proven.*

  Natural deduction is complete. This means that for any statement that is true, i.e. the statement is a correct semantic entailment, there exists a proof.

4.2.2 How can you show that a sequent is not valid? Is this a consequence of soundness or completeness. Explain your answer.

Solution

In order to show that a sequent is not valid, we provide a *counter example*, which is a model that satisfies all premises but falsifies the conclusion.
This is a consequence of soundness. We know from the definition of soundness that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \nvDash \psi \qquad \Rightarrow \qquad \varphi_1, \varphi_2, \ldots, \varphi_n \nvdash \psi$$

A counterexample is enough to tell us that the left-hand side of this implication is true, hence the sequent is not valid.

4.2.3 Explain what it means that natural deduction for propositional logic is *sound*. What is the difference to *completeness*?

Solution

There is no solution available for this question yet.

4.2.4 Look at the following statements and tick them if they are true.

- ☐ Any sequent that is a correct semantic entailment can be proven.
- ☐ Any sequent that can be proven is a correct semantic entailment.
- ☐ If a sequent is not provable, the semantic entailment relation does hold.
- ☐ If for a sequent the semantic entailment relation does not hold, it cannot be proven with natural deduction.

4.2.5 Given an invalid sequent, how do you show its invalidity?

Solution

There is no solution available for this question yet.