Questionnaire “Logic and Computability”

Summer Term 2023

Contents

8 Natural Deduction for Predicate Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

8.1 Natural Deduction Rules

8.1.1 $\forall x\ \big(P(x) \to Q(x)\big), \forall x\ P(x)\ \vdash\ \forall x\ Q(x)$.

8.1.2 $\forall x\ P(x) \land \forall x\ (P(y) \to Q(x)) \quad \vdash \quad Q(z)$

8.1.3 $\forall x(P(x) \land Q(x))\ \vdash\ \forall x P(x) \land \forall x Q(x)$

8.1.4 $\forall x\ P(x) \lor \forall x\ Q(x) \quad \vdash \quad \forall y\ (P(y) \lor Q(y))$

8.1.5 $\forall x\ (P(x) \to Q(y)), \forall y\ (P(y) \land R(x)) \quad \vdash \quad \exists x\ Q(x))$

8.1.6 $\exists x\ \neg P(x), \forall x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$

8.1.7 Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \to Q(x)), \quad \exists x\ P(x) \quad \vdash \quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\forall x\ (P(x) \to Q(x))$	prem.
2.	$\exists x\ P(x)$	prem.
3.	x_0	
4.	$P(x_0)$	ass.
5.	$P(x_0) \to Q(x_0)$	$\forall$e 1
6.	$Q(x_0)$	$\to$ e, 4,5
7.	$\forall x\ Q(x)$	$\forall$i 4-6
8.	$\forall x\ Q(x)$	$\exists$e 2,3-7

8.1.8 $\exists x\ (P(x) \to Q(y)), \quad \forall x\ P(x) \quad \vdash \quad Q(y)$

8.1.9 $\forall x\ \neg(P(x) \land Q(x)) \quad \vdash \quad \neg\exists x\ (P(x) \land Q(x))$

8.1.10 $\exists x\ \neg P(x), \exists x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$

8.1.11 $\exists x\ (P(x) \to Q(y)), \quad \exists x\ P(x) \quad \vdash \quad Q(y)$

8.1.12 $\forall x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \lor Q(x))$

8.1.13 $\forall x\ (P(x) \lor Q(x)), \quad \forall x\ (\neg P(x)) \quad \vdash \quad \forall x\ (Q(x))$

8.1.14 $\neg\exists x\ Q(x) \quad \vdash \quad \forall x\ \neg Q(x)$

8.1.15 $\neg\exists x\ P(x) \lor \neg\exists y\ Q(y)\ \vdash\ \forall z\ \neg(Q(z) \land P(z))$

8.1.16 $\exists x\ (Q(y) \to P(x)) \quad \vdash \quad Q(y) \to \exists x\ P(x)$

8.1.17 $\exists x\ (P(x) \to Q(x)),\ \neg Q(z)\ \vdash\ \neg P(z)$

8.1.18 $\exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ P(x) \land \exists x\ Q(x)$

8.1.19 $\exists x\ (P(x) \lor Q(x)) \quad \vdash \quad \exists x\ P(x) \lor \exists x\ Q(x)$

8.1.20 Explain the $\forall$-introduction rule and the $\forall$-elimination rule. Explain why one rule needs a box while the other one does not. What does it mean that x_0 needs to be fresh?

8.1.21 $\forall x\ (P(x) \land Q(x)) \quad \vdash \quad \forall x\ ((Q(x) \lor R(x)) \land (R(x) \lor P(x)))$

8.1.22 $\exists x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \land R(x))$

8.1.23 $\forall x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \land R(x))$

8.1.24 $\neg\exists x \forall y\ (P(x) \land Q(y)) \quad \vdash \quad \forall x \exists y\ \neg(P(x) \land Q(y))$

8.1.25 $\forall x \exists y\ \neg(P(x) \land Q(y)) \quad \vdash \quad \neg\exists x \forall y\ (P(x) \land Q(y))$

8.1.26 $\neg\exists x\ \neg P(x) \quad \vdash \quad \forall x\ \neg P(x)$

8.1.27 $P(x) \lor Q(y),\quad P(x) \to R(z),\quad Q(y) \to R(z) \quad \vdash \quad R(z)$

8.1.28 $\exists y \forall x\ (P(x,y)) \quad \vdash \quad \forall x \exists y\ (P(x,y))$

8.1.29 $\exists a \forall b\ (S(b,a) \land T(b,a)) \quad \vdash \quad \forall b \forall a\ (S(b,a) \land T(b,a))$

8.1.30 $P(y) \to \forall x Q(x),\ \exists x \neg Q(x)\ \vdash\ \exists x \neg P(x)$

8.1.31 Consider the following natural deduction proof for the sequent

$$\exists x\ \neg P(x) \quad \vdash \quad \neg\forall x\ P(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\exists x\ \neg P(x)$	prem.
2.	$\forall x\ P(x)$	ass.
3.	$P(x_0)$	$\forall$e 2
4.	$\exists x\ P(x)$	$\exists$i 3
5.	$\bot$	$\neg$e 1,4
6.	$\neg\forall x\ P(x)$	$\neg$e 2-5

8.1.32 Consider the following natural deduction proof for the sequent

$$\exists x\ P(x) \lor \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \lor Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\exists x\ P(x) \lor \exists x\ Q(x)$	prem.
2.	$\exists x\ P(x)$	ass.
3.	x_0 $P(x_0)$	ass.
4.	$P(x_0) \lor Q(x_0)$	$\lor$i$_1$ 3
5.	$\exists x\ (P(x) \lor Q(x))$	$\exists$e 2,3-4
6.	$\exists x\ Q(x)$	ass.
7.	x_0 $Q(x_0)$	ass.
8.	$P(x_0) \lor Q(x_0)$	$\lor$i$_2$ 7
9.	$\exists x\ (P(x) \lor Q(x))$	$\exists$e 6,7-8
10.	$\exists x\ (P(x) \lor Q(x))$	$\lor$e 1,2-5,6-9

8.1.33 $\forall x \exists y \; (P(x) \to Q(y)), P(s) \quad \vdash \quad \exists x \forall y \; (\neg P(x) \lor Q(y))$

8.1.34 $\forall a \forall b \; (P(a) \land Q(b)) \quad \vdash \quad \forall a \exists b \; (P(a) \lor Q(b))$

8.1.35 $\exists x \; \neg P(x) \quad \vdash \quad \neg \forall x \; P(x).$