Questionnaire “Logic and Computability”

Summer Term 2024

Contents

11 Soundness and Completeness of Natural Deduction

11.1 Soundness and Completeness of Natural Deduction

11.1.1

(a) Explain what it means that natural deduction for propositional logic is *sound*?

(b) The soundness of natural deduction for propositional logic can be proven via a mathematical induction proof. Discuss the proof idea by stating the *induction hypothesis*, and what needs to be shown for the *induction base-case* as well as for the *induction step*.

11.1.2 The soundness of natural deduction for propositional logic can be proven via a *mathematical course-of-values induction on the length of the Natural Deduction proof.* Let $M(k)$ be the following assertion:

$M(k) :=$ „For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ which have a proof of length k, it is the case that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds."

Your tasks:

(a) Proof the induction base-case, i.e., $M(1)$ holds.

(b) Explain the proof idea of the induction step: $M(1) \wedge M(2) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$.

11.1.3 The soundness of natural deduction for propositional logic can be proven via a *mathematical course-of-values induction on the length of the Natural Deduction proof.* Let $M(k)$ be the following assertion:

$M(k) :=$ „For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ which have a proof of length k, it is the case that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds."

Discuss the **induction step** $M(1) \wedge M(2) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ under the assumption that the $\neg_e$ was applied as a final rule to prove the conclusion.

11.1.4 The soundness of natural deduction for propositional logic can be proven via a *mathematical course-of-values induction on the length of the Natural Deduction proof.* Let $M(k)$ be the following assertion:

$M(k) :=$ „For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ which have a proof of length k, it is the case that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds."

Discuss the **induction step** $M(1) \wedge M(2) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ under the assumption that the $\wedge_i$ was applied as a final rule to prove the conclusion.

11.1.5

(a) Explain what it means that natural deduction for propositional logic is *complete*?

(b) Sketch the idea of the *proof for completeness* of natural deduction for propositional logic. List the three sub-proofs that need to be performed to prove completeness and briefly discuss the idea of each individual sub-proof.

11.1.6 The proof of completeness for natural deduction for propositional logic contains the following sub-proof:

$$\text{If } \models \varphi \text{ holds, then so does } \vdash \varphi.$$

Discuss the proof idea of performing this step of the completeness-proof.