The Laguerre polynomials $L_n(x)$ are defined through

$$L_n(x) = e^x \cdot \left(\frac{d}{dx}\right)^n (x^n e^{-x})$$

Theorem:

$L_n(x)$ satisfies the recurrence relation

$$L_0(x) = 1$$

$$L_{n+1}(x) = (2n + 1 - x) \cdot L_n(x) - n^2 \cdot L_{n-1}(x)$$

for $n \geq 0$ and the differential equation $x \cdot L_n''(x) + (1-x) \cdot L_n'(x) + n \cdot L_n(x) = 0$ for all $n \geq 0$.

Proof:

Let $F := \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} \cdot z^n$ be the exponential generating function of the sequence of polynomials. It is the diagonal series of the power series

$$G := \sum_{m,n=0}^{\infty} \frac{1}{m!} \cdot e^x \cdot \left(\frac{d}{dx}\right)^m (x^n e^{-x}) \cdot y^m \cdot z^n$$

Because the Taylor series development theorem holds in formal power series rings (see [1], section 2.16), we can simplify

$$\begin{aligned} G &= e^x \cdot \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(\frac{d}{dx}\right)^m (x^n e^{-x}) \cdot y^m \right) \cdot z^n \\ &= e^x \cdot \sum_{n=0}^{\infty} (x+y)^n e^{-(x+y)} \cdot z^n \\ &= \frac{e^{-y}}{1 - (x+y)z} \end{aligned}$$

We take over the terminology from the "diag_rational" paper; here $R = Q[x]$ and $M = Q[[x]]$ (or, if you like it better, $M = H(C)$, the algebra of functions holomorphic in the entire complex plane). $G \in M[[y, z]]$ is not rational; nevertheless we can proceed similarly to the "diag_series" paper. $F(z^2)$ is the coefficient of t^0 in

$$G(zt, \frac{z}{t}) = \frac{e^{-zt}}{1 - z^2 - \frac{xz}{t}} \in M[[zt, \frac{z}{t}, z]] = M\langle\langle z, t\rangle\rangle$$

The denominator's only zero is $t = \frac{xz}{1-z^2}$. We can write

$$e^{-zt} = e^{-\frac{xz^2}{1-z^2}} + \left(zt - \frac{xz^2}{1-z^2} \right) \cdot P(z,t)$$

with $P(z,t) \in Q[[zt, \frac{xz^2}{1-z^2}]] \subset Q[[zt, x, z]] = M[[zt, z]] \subset M\langle\langle z, t\rangle\rangle$. This yields – all computations being done in $M\langle\langle z, t\rangle\rangle$ –

$$\begin{aligned} G(zt, \frac{z}{t}) &= \frac{e^{-\frac{xz^2}{1-z^2}}}{1 - z^2 - \frac{xz}{t}} + \frac{zt}{1-z^2} \cdot P(z,t) \\ &= \frac{1}{1-z^2} \cdot e^{-\frac{xz^2}{1-z^2}} \cdot \sum_{j=0}^{\infty} \left(\frac{x}{1-z^2} \frac{z}{t} \right)^j + \frac{zt}{1-z^2} \cdot P(z,t) \end{aligned}$$

Here, the coefficient of t^0 is

$$F(z^2) = \frac{1}{1-z^2} \cdot e^{-\frac{xz^2}{1-z^2}}$$

hence

$$F(z) = \frac{1}{1-z} \cdot e^{-\frac{xz}{1-z}}$$

It follows that $(1-z)^2 \cdot \frac{d}{dz} F - (1 - x - z) \cdot F = 0$. This is equivalent to the claimed recurrence.

Starting from the closed form for F, we compute a linear relation for the partial derivatives of F. Write $\partial_x = \frac{d}{dx}$ and $\Delta_z = z\frac{d}{dz}$. One computes

$$F = 1 \cdot F$$

$$(1-z) \cdot \partial_x F = -z \cdot F$$

$$(1-z)^2 \cdot \partial_x^2 F = z^2 \cdot F$$

$$(1-z)^2 \cdot \Delta_z F = ((1-x)z - z^2) \cdot F$$

$$(1-z)^3 \cdot \partial_x \Delta_z F = (-z + xz^2 + z^3) \cdot F$$

Solve a homogeneous 4×5 system of linear equations over $Q(x)$ to get

$$(1-z)^3 \cdot \Big((1-x) \cdot \partial_x F + x \cdot \partial_x^2 F + \Delta_z F \Big) = 0$$

Divide by the first factor to get

$$(1-x) \cdot \partial_x F + x \cdot \partial_x^2 F + \Delta_z F = 0$$

This is equivalent to the claimed equation $x \cdot L_n''(x) + (1-x) \cdot L_n'(x) + n \cdot L_n(x) = 0$.

References

1 Bruno Haible: D-finite power series in several variables. *Diploma thesis, University of Karlsruhe, June 1989.* Sections 2.15 and 2.22.